Overflow

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Abstract

This paper re-interprets the concept of unbounded intervals in terms of overflow. All of the "mathematical intervals" at Level 1 are defined to be closed and bounded; and the concept of overflow is then introduced in a new "intermediate" level between Level 1 and Level 2. For the arithmetic operations in particular, this new structural interpretation is functionally equivalent to the current model of unbounded intervals. The midpoint of an unbounded interval is undefined, however, and therefore leads to interval algorithms that generate NaNs; but midpoint can be defined as a real number for overflow, thereby providing algorithms that generate useful results.

1 Introduction

The following is from [1]:

> A closed interval is an interval that includes all of its limit points. If the endpoints of the interval are finite numbers a and b, then the interval $\{x : a \leq x \leq b\}$ is denoted $[a, b]$. If one of the endpoints is $\pm\infty$, then the interval still contains all of its limit points (although not all of its endpoints), so $[a, \infty)$ and $(-\infty, b]$ are also closed intervals, as is the interval $(-\infty, \infty)$.

If $\mathbb{R}$ is the set of real numbers and

$$\mathbf{I}\mathbb{R} \equiv \{[a, b] : a, b \in \mathbb{R} \wedge a \leq b\}$$

is the classic set of nonempty, closed and bounded intervals, then by P1788 conventions $\overline{\mathbf{I}\mathbb{R}}$ is the extension of $\mathbf{I}\mathbb{R}$ to the set of closed intervals [1], including the empty set. Intervals of the form $[a, +\infty]$, $[-\infty, b]$ and $[-\infty, \infty]$ are therefore understood to be closed, unbounded intervals. Note the use of square brackets, i. e., in all cases infinity is not considered to be an element of any interval. The singletons $[-\infty, -\infty]$ and $[+\infty, +\infty]$ are by definition invalid constructions.

Arithmetic operations [7] on the endpoints of unbounded intervals follow conventions of real analysis and IEEE 754 arithmetic such as

$$\begin{aligned}(-\infty)+(-\infty) &= -\infty \\ (+\infty)+(+\infty) &= +\infty \\ (\pm\infty)\cdot(\pm\infty) &= +\infty \\ (\pm\infty)\cdot(\mp\infty) &= -\infty \\ \forall x\in\mathbb{R},\ x/(\pm\infty) &= 0.\end{aligned}$$

The operations $(-\infty)+(+\infty)$ and $(+\infty)+(-\infty)$ on the endpoints of unbounded intervals [5] are exceptional conditions, as in IEEE 754 arithmetic. Presumably, a future motion for inner multiplication and division (or Kaucher arithmetic) would likewise define operations involving ratios of infinities on the endpoints of unbounded intervals as exceptional conditions, which would also be consistent with IEEE 754 arithmetic.

One point of deviation from IEEE 754 is that [7] defines $0\cdot(\pm\infty) = 0$ and $(\pm\infty)\cdot 0 = 0$. These operations are exceptional conditions in IEEE 754 arithmetic, but in the context of P1788 the infinity is understood to be the accumulation point of an operand representing a set of real numbers in an unbounded interval arithmetic operation, hence the non-exceptional result.

The purpose of this paper is not to overthrow any of these conventions or to argue they are invalid (we believe they are correct); the purpose is to re-interpret the concept of unbounded intervals in terms of overflow. This is accomplished by defining the "mathematical intervals" at Level 1 as $\mathbf{I}\mathbb{R}$, and then introducing the concept of overflow in a new "intermediate" level between Level 1 and Level 2. For the arithmetic operations in particular, this new structural interpretation is functionally equivalent to the current model of unbounded intervals. The midpoint of an unbounded interval is undefined, however, and leads to interval algorithms that generate NaNs; but midpoint can be defined as a real number for overflow, thereby providing algorithms that generate useful results.

2 Level Structure

In this paper, the Level 1 set of "mathematical intervals" is defined as $\mathbf{I}\mathbb{R}$. This gives the classic Fundamental Theorem of Interval Arithmetic (FTIA) of Ramon Moore, where the natural interval extension of a real function is defined only if the interval input is a nonempty subset of the natural domain of the real function; and any result satisfying the theorem is a nonempty, closed and bounded interval. The infimum, supremum, midpoint and radius of a Level 1 interval is always defined and is always a real number.

Level 2 is mostly unchanged in this paper. Each Level 2 interval format is associated with some finite subset of the reals. The number of elements in the subset is finite, and each element is a real number (the issue of infinity vs. overflow as members of this set will be discussed later). The maximal real element of the finite set is `Fmax`.

The concept of overflow is introduced in a new "intermediate" level between Level 1 and Level 2. We call this Level 1a. In the current P1788 model, this is where arithmetic in $\mathbf{I}\mathbb{R}$ would otherwise be extended to the unbounded intervals and the empty set of $\overline{\mathbf{I}\mathbb{R}}$. Instead of doing this, however, we re-interpret the arithmetic of unbounded intervals in terms of overflow.

2.1 Level 1: Mathematical Intervals

The "mathematical intervals" are defined as the set $\mathbf{I}\mathbb{R}$ of nonempty, closed and bounded intervals. If $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a real function, $D_f \subseteq \mathbb{R}^n$ is the natural domain of f, and $X \in \mathbf{I}\mathbb{R}^n$ is a nonempty subset of D_f, then the interval hull of the set

$$\{f(x) : x \in X\} \tag{1}$$

is simply the minimum and maximum of f over the interval domain X; so we may define the natural interval extension

$$f(X) \equiv [\min_{x \in X} f(x), \max_{x \in X} f(x)] \tag{2}$$

as a function $f(X) : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}$. This gives the classic Fundamental Theorem of Interval Arithmetic (FTIA) of Ramon Moore, where $F(X) \in \mathbf{I}\mathbb{R}$ is any interval that satisfies

$$f(X) \subseteq F(X) \Leftrightarrow (\forall x \in X)(\exists y \in F(X)) : y = f(x). \tag{3}$$

If $[a, b] \in \mathbf{I}\mathbb{R}$ is a mathematical interval, then

$$\begin{aligned} inf([a,b]) &\equiv a \\ sup([a,b]) &\equiv b \\ mid([a,b]) &\equiv (a+b)/2 \\ rad([a,b]) &\equiv (b-a)/2 \end{aligned}$$

is always defined and is always a real number.

2.2 Level 1a: Overflow

We now introduce the concept of overflow in a new "intermediate" level between Level 1 and Level 2. We call this Level 1a. With respect to the current P1788 model, Level 1a is where arithmetic in $\mathbf{I}\mathbb{R}$ would be extended to the unbounded intervals and the empty set of $\overline{\mathbf{I}\mathbb{R}}$. Instead of doing this, however, we re-interpret the arithmetic of unbounded intervals in terms of overflow.

Level 1a may be thought of as an abstract or "virtual" parameterization of Level 2. We are at Level 1 in the sense we are still working on the real number line with an infinite amount of precision, however we also introduce an overflow threshold h onto the real number line. Any real number x such that $x > h$ or $x < -h$ is considered "overflow" at Level 1a. In this paper, we use $+\omega$ and $-\omega$, respectively, as symbols for positive and negative overflow.

The threshold value h is a virtual parameterization of `Fmax`. If there will be multiple Level 2 formats, there may be multiple virtual parameterizations $h_0, h_1, \ldots, h_n$ at Level 1a for each `Fmax` at Level 2. This is discussed later in the paper. For now we just assume there is one Level 2 format and hence one virtual parameter h at Level 1a.

FTIA (3) can be extended into $\overline{\mathbf{I}\mathbb{R}}$ by re-defining the set (1) as

$$\{f(x) : x \in X_f\}, \text{ with } X_f = X \cap D_f \tag{4}$$

for any $X \in \overline{\mathbf{I}\mathbb{R}}^n$. Similarly, (2) can be re-defined as

$$f(X) \equiv [\inf_{x \in X_f} f(x), \sup_{x \in X_f} f(x)]. \tag{5}$$

Note that (4) and (5) are the current Level 1 definitions given in the draft standard text [10].

Exercise 1 *What exactly is overflow? Let's build intuition and start with something familiar. With (4) and (5) we may consider a function like $g(x) = 1/x$ on the interval domain $X = [0, 1]$. The natural interval extension in this case is the closed, unbounded interval $[1, +\infty]$. Let the overflow threshold be $h = 100$. Note that the interval*

$$[1, +\infty] \equiv \{x : x \geq 1\}$$

contains real numbers x such that $x > 100$. In Level 1a the interval $[1, +\infty]$ is therefore considered to "overflow" into a family of intervals

$$[1, +\omega] \equiv \{[1, \delta] : \delta \geq 100\}.$$

For the sake of intuition, there are a few important facts to notice from the example in Exercise 1:

- Note that "overflow" is not an interval, it is a family of intervals.
- Note that due to the overflow threshold $h = 100$ the interval $[1, 100]$ is the least of all intervals in the family of intervals; it is also the only interval in the family of intervals that is a subset of $[-h, h]$.
- Note that all intervals in the family of intervals are nonempty, closed and bounded "mathematical intervals," i. e., they are all elements of $\mathbf{I}\mathbb{R}$.
- Note that the union of all intervals in the family of intervals is the closed, unbounded interval $[1, +\infty]$.

These facts illustrate overflow as a re-interpretation of unbounded intervals, since the mapping of an unbounded interval into a family of intervals parameterized by some overflow threshold h is always well-defined. Similarly, the union of all intervals in this family of intervals is always an unbounded interval.

Definition 1 (Overflow Family) *If* $X = [a, b]$ *is a nonempty element of* $\overline{\mathbf{I}\mathbb{R}}$ *and* $h \in \mathbb{R}$ *is an overflow threshold such that* $H = [-h, h]$*, then the overflow family of* X *is defined*

$$\Omega(X) \equiv \begin{cases} [-\omega, -h] & \text{if } X \prec H \\ [-\omega, b] & \text{if } X \not\prec H \wedge X < H \\ \{X\} & \text{if } X \subseteq H \\ [-\omega, +\omega] & \text{if } H \subset X, \text{ i. e., } H \text{ is interior to } X \\ [a, +\omega] & \text{if } H \not\prec X \wedge H < X \\ [h, +\omega] & \text{if } H \prec X \end{cases} \tag{6}$$

where any overflow family of the form $[-\omega, v]$*,* $[u, +\omega]$ *or* $[-\omega, +\omega]$ *with* $u, v \in H$ *is defined*

$$[-\omega, v] \equiv \{[-\delta, v] : \delta \in \mathbb{R} \wedge \delta \geq h\}, \tag{7}$$

$$[u, +\omega] \equiv \{[u, \delta] : \delta \in \mathbb{R} \wedge \delta \geq h\}, \tag{8}$$

$$[-\omega, +\omega] \equiv \{[-\delta, \delta] : \delta \in \mathbb{R} \wedge \delta \geq h\}. \tag{9}$$

The overflow family of the empty set, i. e., $\Omega(\varnothing)$*, is the singleton* $\{\varnothing\}$*.*

Corollary 1 *If for any* $X \in \overline{\mathbf{I}\mathbb{R}}$ *we define* $\Upsilon(Z)$ *as the union of all intervals in the overflow family* $Z = \Omega(X)$*, then*

$$\Upsilon(\Omega(X)) : \overline{\mathbf{I}\mathbb{R}} \to \overline{\mathbf{I}\mathbb{R}}. \tag{10}$$

Corollary 2 *The mapping (10) has the property*

$$X \subseteq \Upsilon(\Omega(X)). \tag{11}$$

Proposition 1 *For any* $X \in \overline{\mathbf{I}\mathbb{R}}^n$*,* $F(X) \in \overline{\mathbf{I}\mathbb{R}}$ *and natural interval extension* $f(X)$ *as defined in (5),*

$$f(X) \subseteq F(\Upsilon(\Omega(X))) \Leftrightarrow (\forall x \in X)(\exists y \in F(\Upsilon(\Omega(X)))) : y = f(x).$$

Proof. *If* $f(X) \subseteq F(X)$*, then*

$$F(X) \subseteq F(\Upsilon(\Omega(X))) \Rightarrow f(X) \subseteq F(\Upsilon(\Omega(X))).$$

■

Proposition 2 *For any* $X \in \overline{\mathbf{I}\mathbb{R}}^n$*,* $F(X) \in \overline{\mathbf{I}\mathbb{R}}$ *and natural interval extension* $f(X)$ *as defined in (5),*

$$f(X) \subseteq \Upsilon(\Omega(F(X))) \Leftrightarrow (\forall x \in X)(\exists y \in \Upsilon(\Omega(F(X)))) : y = f(x).$$

Proof. If $f(X) \subseteq F(X)$, then

$$F(X) \subseteq \Upsilon(\Omega(F(X))) \Rightarrow f(X) \subseteq \Upsilon(\Omega(F(X))).$$

■

Theorem 1 (Overflow Arithmetic) *If $F(X)$ is an interval extension of a real function over the domain $X \in \overline{\mathbb{IR}}^n$, then for any overflow family Z we may safely re-interpret $F(X)$ as*

$$\Omega(F(\Upsilon(Z))).$$

Exercise 2 *Consider the real function*

$$f(x) = 1 + x^2(1 + \sin(x))$$

and suppose we want proof of the non-existence of any solution $f(x) = 0$ on the domain of the overflow family $Z = [-\omega, +\omega]$. By Theorem 1 we have

$$\begin{aligned} \Omega(f(\Upsilon(Z))) &= \Omega(f([-\infty, +\infty])) \\ &= \Omega([1, +\infty]) \\ &= [1, +\omega]. \end{aligned}$$

Zero is not an element of any interval in the overflow family $[1, +\omega]$, nor is it an element of the unbounded interval $\Upsilon([1, +\omega]) = [1, +\infty]$ which represents the union of all intervals in the overflow family. Under either interpretation, it is proof of non-existence of the solution.

2.2.1 Multiple Parameterizations

If there will be multiple Level 2 formats, there may be multiple virtual parameterizations $h_0, h_1, \ldots, h_n$ at Level 1a for each `Fmax` at Level 2. We can similarly define respective mappings $\Omega_0, \Omega_1, \ldots, \Omega_n$. Conversion, for example, between parameterizations h_0 and h_1 is then easily defined, i. e., if $X_0 = \Omega_0(X)$, then $X_1 = \Omega_1(\Upsilon(X_0))$.

2.3 Level 2: Interval Datums

Each Level 2 interval format is associated with some finite subset of the reals. The number of elements in the subset is finite, and each element is a real number. The set is then augmented by the two elements $+\omega$ and $-\omega$, respectively, as symbols for positive and negative overflow. The maximal real element of the augmented set is `Fmax`, and this is the concrete value assigned to the virtual overflow threshold h in the Level 1a parameterization of the Level 2 format.

3 Rationale

This section illustrates how overflow makes available new opportunities to clarify or "fix" some aspects of the existing P1788 model (and perhaps even IEEE 754) that are arguably ambiguous, inconsistent or correct but otherwise somehow unexpected or problematic.

3.1 IEEE 754 and P1788 Arithmetic Operations

For the interval arithmetic operations in particular and FTIA in general, overflow is functionally equivalent to the current model of unbounded intervals. Assuming that IEEE 754 infinities are used at Level 3 to represent overflow, all of the arithmetic operations in [7] on the endpoints of unbounded intervals are unchanged when re-interpreted as overflow, i. e.,

$$\begin{aligned}
(-\omega) + (-\omega) &= -\omega \\
(+\omega) + (+\omega) &= +\omega \\
(\pm\omega) \cdot (\pm\omega) &= +\omega \\
(\pm\omega) \cdot (\mp\omega) &= -\omega \\
\forall x \in \mathbb{R},\ x/(\pm\omega) &= 0.
\end{aligned}$$

The operations $(-\omega) + (+\omega)$ and $(+\omega) + (-\omega)$ in Kaucher arithmetic would represent exceptional conditions, as would endpoint operations involving ratios of overflow; all of this is consistent with IEEE 754 arithmetic if the infinities are interpreted as overflow.

Note that when the unbounded intervals are re-interpreted as overflow, we still have $0 \cdot (\pm\omega) = 0$ and $(\pm\omega) \cdot 0 = 0$, as in [7].

One historical issue is that when the IEEE 754 standard was created, it overloaded the concept of infinity and overflow. The affine infinities $+\infty$ and $-\infty$ are defined as true infinities, and arithmetic operations are defined accordingly. This is why, for example, $0 \cdot (\pm\infty)$ and $(\pm\infty) \cdot 0$ are exceptional operations in IEEE 754 arithmetic. But $+\infty$ and $-\infty$ may also be the result of certain operations that overflowed. In this case, the infinite result only represents a very large but finite and unrepresentable floating-point number. This issue has been and continues to be a point of controversy. With the benefit of hindsight, if the original IEEE 754 level structure had introduced the concept of overflow into a Level 1a similar to the one described in this paper it may have been possible to avoid this problem.

In the event some future revision of IEEE 754 might introduce the concept of overflow so that $+\infty$ and $-\infty$ are differentiated from $+\omega$ and $-\omega$, and if

$$\begin{aligned}
0 \cdot (\pm\infty) &= \texttt{NaN} \\
(\pm\infty) \cdot 0 &= \texttt{NaN} \\
0 \cdot (\pm\omega) &= 0 \\
(\pm\omega) \cdot 0 &= 0
\end{aligned}$$

the interval arithmetic would be compatible. Vendors could use $+\omega$ and $-\omega$ at Level 3 to represent overflow. Without such a future revision of IEEE 754, of course, vendors can use $+\infty$ and $-\infty$ at Level 3 to represent overflow, taking into consideration the necessary differences mentioned above and in [7].

$M([a,b])$	$b=-\omega$	$b\in H$	$b=+\omega$
$a=-\omega$	$-\omega$	$-h$	0
$a\in H$	$-h$	$M(a,b)$	$+h$
$a=+\omega$	0	$+h$	$+\omega$

Table 1: For any overflow family $[a, b]$ parameterized by an overflow threshold $h \in \mathbb{R}$, the bisection method $M([a, b])$ may be defined at Level 1a as shown in this table. $H = [-h, h]$, and $M(a, b)$ is a real number computed by the bisection method M as a function of $a, b \in H$.

3.2 Mean, Median and Other Bisection Methods

Interval bisection is fundamental to interval algorithms. There are many ways to bisect an interval, but any bisection method M must compute a point p as a function of an input interval $[a, b]$ such that $p \in [a, b]$ may form the two intervals $[a, p]$ and $[p, b]$.

Computing the midpoint of $[a, b]$ is perhaps the simplest bisection method, since it is simply the arithmetic mean of the interval endpoints. One might also consider the geometric mean or even more sophisticated methods. In [11], John Pryce defines, e. g.,

$$smedian2([a,b]) = sinh\left(\frac{asinh(a) + asinh(b)}{2}\right). \tag{12}$$

The midpoint of an unbounded interval is undefined. This fact has prompted several prominent members of P1788 to suggest the midpoint of an unbounded interval should be undefined at Level 2, as well, and therefore return NaN. We agree with this position. In fact, we believe the more general bisection methods described above are undefined for an unbounded interval, too (even at Level 2), and should similarly return NaN.

However, this means many interval algorithms that use bisection methods are undefined at both Level 1 and Level 2 in the current model. For example, how may a branch-and-bound algorithm that bisects on the midpoint even begin to proceed (at Level 1 or Level 2) if the user provides $X = [1, +\infty]$ as input?

In this paper,

$$Z = \Omega(X) = \Omega([1, +\infty]) = [1, +\omega]$$

is an overflow family. This is a different mathematical object than an unbounded interval, so we may define the bisection point of $[1, +\omega]$ as a real number. Table 1 shows one way this may be accomplished.

Now all the interval algorithms that are undefined for unbounded intervals in the current model are defined for overflow in the new model. By re-interpreting unbounded intervals as overflow, such interval algorithms may return useful results instead of NaNs.

3.2.1 Is Overflow Really a Novel Concept?

Some discussion in P1788 has objected to overflow on the premise it is a new idea that is not well understood. But is that really true?

All vendors and implementers of interval libraries have at some time been faced with the problem of trying to define a bisection method (arithmetic mean, geometric mean, median, etc.) for an unbounded interval such as $[1, +\infty]$. How has this been dealt with in the past?

Such bisection methods are invariably undefined at Level 1, but special cases and exceptional conditions at Level 2 are typically handled along the lines of the rules illustrated for overflow in Table 1. A case in point is the smedian2 bisection method found in [11]. This method is undefined at Level 1 when the input is unbounded, but in the source code of the reference implementation, the author leaves the following comment about the implementation at Level 2:

```
% (C) John Pryce 2012. Thanks to Dan Zuras for his analysis.
% M = SMEDIAN2(A,B) computes sinh(0.5*(asinh(a)+asinh(b)))
% - The case when either of A,B is infinite is treated specially,
%   as shown in this table, where H denotes the largest
%   representable real, REALMAX:
%        \ b=
%      a= \   -inf    finite    +inf  |
%   -------+--------+--------+--------+
%    -inf  |  -inf     -H        0    |
%   finite |   -H      n/a       +H   |
%    +inf  |    0      +H       +inf  |
%   -------+--------+--------+--------+
```

In this comment, we see that the Level 2 definition depends on the notion of "the largest representable real, REALMAX." The table that follows in the source code comment is practically identical to Table 1.

One must consider the logical implications. If one postulates the arithmetic mean (midpoint) of an unbounded interval is undefined at Level 2, it follows that the geometric mean of an unbounded interval should also be undefined at Level 2. Since smedian2 is a variant of geometric mean, why is smedian2 defined at Level 2 for an unbounded interval but geometric and arithmetic means are not? In our view, this represents a logical contradiction and illustrates the perils of ad-hoc definitions and special-case reasoning.

On the other hand, this also illustrates why we believe that overflow is not a new concept. One simply must be able to define bisection methods in these cases, and overflow provides the necessary framework to do so without ending up in the snare of logical contradictions just mentioned. This is due to the fact that an overflow family is a different mathematical object than an unbounded interval because an overflow family is parameterized by a threshold representing "the largest representable real, REALMAX" but an unbounded interval is not.

We can therefore define bisection methods on overflow, and this avoids the contradiction that the same methods are also undefined for unbounded intervals.

Furthermore, we can define the bisection methods at both Level 1a as well as Level 2 for overflow, an accomplishment that simply is not at all possible in the current P1788 model.

3.2.2 Interval Newton

Perhaps one of the most fundamental and important algorithms is the extended Interval Newton. At the heart of the algorithm is the Newton operator

$$N(X) \equiv x - \frac{f(x)}{f'(X)}, \text{ with } x \in X. \tag{13}$$

But (13) is a mathematical definition. To be an algorithm, an exact method of choosing $x \in X$ must be specified. A practical, simple and easy choice that works well for a large class of problems is to choose the midpoint of X, i. e.,

$$N_{mid}(X) \equiv mid(X) - \frac{f(mid(X))}{f'(X)}. \tag{14}$$

Now (14) is an algorithm. But users may be surprised to learn that if X is an unbounded interval, (14) is undefined and may generate NaNs.

In the model presented in this paper, bisection on the midpoint in (14) may be defined for overflow according to Table 1. Under this definition, useful results are obtained for the function $f(x) = x^2(2x - 3)$ on the input domain $X_0 = [1.1, +\omega]$:

```
Xn                     mid(Xn)    N(Xn)
=================      =======    =================
[    1.1, 1.#INF]    1.7977e+308    [-1.#INF,1.7977e+308]
[    1.1,1.7977e+308]    8.9885e+307    [-1.#INF,8.9885e+307]
[    1.1,8.9885e+307]    4.4942e+307    [-1.#INF,4.4942e+307]
[    1.1,4.4942e+307]    2.2471e+307    [-1.#INF,2.2471e+307]
...
[    1.1, 10.096]     5.5981    [-383.58, 5.1319]
[    1.1, 5.1319]      3.116    [-44.429, 2.8693]
[    1.1, 2.8693]     1.9847    [-3.8003,  1.866]
[    1.1,  1.866]      1.483    [ 1.4907, 1.5962]
[ 1.4907, 1.5962]     1.5435    [ 1.4963, 1.5072]
[ 1.4963, 1.5072]     1.5017    [    1.5,    1.5]

Solution: [1.49998,1.50003]

Total iterations: 990
```

Other bisection methods such as geometric mean, smedian2, etc. may similarly be used to select $x \in X$ in (13). When provided with an overflow family as input, such interval algorithms may also generate useful results, as opposed to NaNs. For example, if the algorithm uses smedian2 in accordance with Table 1 as the bisection method for the overflow input, the solution is found in only 14 total iterations.

3.2.3 Centered Forms

Similar situations exist when working with centered forms. For example,

$$F(X) \equiv f(c) + (X - c)f'(X), \text{ with } c \in X \tag{15}$$

is a mathematical definition of the mean-value form of $F(X)$; and the method of choosing $c \in X$ is typically the midpoint of X, i. e.,

$$F_{mid}(X) \equiv f(mid(X)) + (X - mid(X))f'(X). \tag{16}$$

As in the case of the Newton operator, (16) is undefined in the current P1788 model when X is an unbounded interval, and it may therefore generate NaNs. The same for various forms of the Krawczyk operator and related functions described in [13], including the Baumann theorem. With overflow, these forms and functions may be defined.

3.2.4 Branch and Bound

In branch-and-bound, an initial interval domain X_0 of a function f is specified by a user and then recursively bisected into

$$X_0 \supset X_1 \cdots \supset X_n$$

until some X_i or $f(X_i)$ satisfies an acceptance, deletion or termination criteria. Bisecting at the midpoint is very common and works extremely well for a large class of interval problems. Users often want to specify the initial interval domain to be as wide as possible, and a convenient way to do this is to specify X_0 as an unbounded interval. However, such an algorithm is not defined in the current P1788 model and will generate NaNs.

Useful results can be obtained by re-interpreting the unbounded intervals as overflow. For example, a simple midpoint bisection algorithm performed on the function $f(x) = \sin(1/x)$ on the input domain $X_0 = [0.1, +\omega]$ finds all four solutions (including the solution at infinity):

```
Xi                      mid(Xi)    F(Xi)
=================       =======    =================
[1.7977e+308, 1.#INF]     1.7977e+308    [       0,5.5627e-309]
[0.31094,0.31875]       0.31484    [-0.074419,0.0043377]
[0.15469, 0.1625]       0.15859    [-0.12898,0.18047]
[     0.1,0.10781]      0.10391    [-0.54402,0.14886]

Total bisections: 1038
```

If the algorithm uses smedian2 on the overflow input in accordance with Table 1 as the bisection method, all four solutions are found in only 26 total bisections. We notice that neither the midpoint nor smedian2 bisection methods are defined in the current model for the unbounded input $X_0 = [0.1, +\infty]$ and that both methods in this case would simply generate NaNs.

3.2.5 Minimal Standards: An Analogy

In computer science, an important utility that digital computer systems should provide is the ability to generate random numbers. Over the course of time, it has become widely known that good random number generators are hard to find. This was the premise of the famous publication [9], which therefore argued for a "minimal standard." This became known as the Park-Miller Pseudo-Random Number Generator (PM-PRNG).

The PM-PRNG is very simple and easy to implement, and it is known to have "good" statistical properties. For many basic or simple applications it is entirely suitable. Experts and specialists, of course, may choose random number generators that are known to have "better" properties (such as DRAND48 or the Mersenne Twister). But for non-experts and non-specialists, the PM-PRNG minimal standard provides users with a random number generator that is known to provide useful and satisfactory results for most non-specialty applications.

We see this as an analogy to bisection methods in interval analysis. There are many ways to bisect an interval, and not all bisection methods are equally suited to the certain or special needs of various interval algorithms. However, we believe there is a need for a "minimal standard" when it comes to interval bisection methods. For us, this is the midpoint bisection method. We therefore believe P1788 should, at a minimum, standardize a midpoint operation on overflow.

3.3 Comparison Operations

It may seem obvious to most people that $[2, 100]$ is not interior to $[1, 100]$ because both intervals share a common upper endpoint of 100. But even experts are sometimes surprised to learn that $[2, +\infty]$ is interior to $[1, +\infty]$.

One might call this "astonishing." But it is the mathematical truth; at least for unbounded intervals. In [12], if A and B are nonempty elements of $\overline{\mathbb{IR}}$, then the boolean function $isInterior(A, B)$ is defined

$$((\forall a \in A)(\exists b \in B) : b < a) \wedge ((\forall a \in A)(\exists b \in B) : a < b) \tag{17}$$

and we have

$$isInterior(A, B) \Leftrightarrow (inf(B) <' inf(A)) \wedge (sup(A) <' sup(B)) \tag{18}$$

where $<'$ is the same as $<$ except that $-\infty <' -\infty$ and $+\infty <' +\infty$ are true.

For (17) and (18), it is easy to see that

$$\begin{aligned} isInterior([2, 100], [1, 100]) &= false, \\ isInterior([2, +\infty], [1, +\infty]) &= true. \end{aligned}$$

The implementation of $isInterior$ necessarily becomes more complex to accommodate the special cases of $<'$ required for unbounded intervals.

All of the definitions in [12] are compatible with overflow and require no changes. If A and B are overflow families instead of unbounded intervals, we may simply define any comparison operation $\circ$ to be

$$\Upsilon(A) \circ \Upsilon(B). \tag{19}$$

For example,

$$\begin{aligned} isInterior([2,+\omega],[1,+\omega]) &= isInterior(\Upsilon([2,+\omega]),\Upsilon([1,+\omega])) \\ &= isInterior([2,+\infty],[1,+\infty]) \\ &= true. \end{aligned}$$

Re-interpreting unbounded intervals as overflow also provides new opportunities to consider that are currently unavailable. For any overflow family Z we could define $\Lambda(Z)$ to be the least interval (by containment order) of all intervals in the overflow family Z; then any comparison operation $\circ$ could be defined

$$\Lambda(A) \circ \Lambda(B). \tag{20}$$

The *isInterior* operator would then give

$$\begin{aligned} isInterior([2,+\omega],[1,+\omega]) &= isInterior(\Lambda([2,+\omega]),\Lambda([1,+\omega])) \\ &= isInterior([2,h],[1,h]) \\ &= false. \end{aligned}$$

In this case an implementation could use the very simple and efficient programming formulas originally presented in [8], so long as the infinities in those formulas were interpreted as overflow. In other words, the special cases associated with $<'$ in [12] would no longer be needed and $<$ could be used instead.

Another possibility for comparison operations on overflow might be to consider multi-valued logic [2]. We prefer boolean comparison operations, however, so have not spent time looking seriously at this option.

3.4 Kaucher Arithmetic

The beginning of Kaucher arithmetic comes from embedding the commutative cancellative monoid $(\mathbf{I}\mathbb{R},+)$ into the abelian group $(\mathbb{I}\mathbb{R},+)$ as described in [4]. The system $(\overline{\mathbf{I}\mathbb{R}},+)$ is not cancellative due to the unbounded intervals and empty set, and this makes it an unsuitable starting point.

In this paper, the concept of unbounded intervals and their reinterpretation as overflow isn't introduced until Level 1a. Kaucher arithmetic can then be developed at Level 1 in a manner compatible to P1788, and extensions of Kaucher arithmetic to unbounded intervals and overflow families could possibly be introduced at Level 1a, etc.

Remark 1 *Even the "cancel minus" or "Hukuhara difference" in section 5.4.5 of the current draft text requires cancellation, and this is why the operation is not defined for unbounded intervals.*

We believe the classic set of nonempty, closed and bounded intervals is a very special set of intervals that deserves its own level in the standard. The Level 1 presented in this paper represents a common intersection between all of the main branches of interval arithmetic, including the classical interval arithmetic of Ramon Moore, the generalized interval arithmetic of Edgar Kaucher [6], and the modal interval analysis of E. Gardenes and the SIGLA/X group [3].

Aside from the other unrelated issues of overflow and bisection methods, we therefore believe the level structure presented in this paper provides a framework for an interval standard that will better endure the test of time.

4 Conclusion

This paper re-interprets unbounded intervals as overflow. A closed, unbounded interval has a well-defined mapping into a family of intervals parameterized by an overflow threshold $h \in \mathbb{R}$, and the union of all intervals in this family is a closed, unbounded interval. This makes it easy to think about unbounded intervals in terms of overflow and vice-versa.

In practice, overflow is functionally equivalent to unbounded intervals in regards to the arithmetic operations and interval functions. Although some mathematical formalities are different at the upper levels, the implementation of overflow at Level 3 requires no changes to the existing operations on unbounded intervals. In this regard, it is largely a matter of semantics. The midpoint of an unbounded interval is undefined, but the midpoint of an overflow family may be defined as a real number. The same is true for other bisection methods such as geometric mean or [11].

Aside from the unrelated issue of overflow, a Level 1 that is defined to be restricted to the nonempty, closed and bounded intervals represents a common point of intersection between all of the main branches of interval arithmetic. We therefore believe the level structure presented in this paper provides a framework for an interval standard that will better endure the test of time.

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