PART 2

Interval Standard (Simplified)

4. Level 1 description

In this clause, subclauses 4.1 to 4.4 describe the theory of set-based intervals and interval functions. Subclause 4.5 lists the required *arithmetic operations* (also called elementary functions) with their mathematical specifications.

4.1 Level 1 entities

Set-based intervals deal with entities of the following kinds.

– The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of **extended reals**.

– The set of **(text) strings**, namely finite sequences of **characters** chosen from some alphabet.

– The set of **integers**.

– The **boolean** values `false` and `true`.

– The set of **decorations** (defined in 5).

Any member of $\overline{\mathbb{R}}$ is called a number. It is a **finite number** if it belongs to $\mathbb{R}$, else an **infinite number**. An interval's members are finite numbers, but its bounds can be infinite. Finite or infinite numbers can be inputs to interval constructors, as well as outputs from operations, e.g., the interval width operation.

Since Level 1 is primarily for human communication, there are no Level 1 restrictions on the alphabet used. Strings may be inputs to interval constructors, as well as inputs/outputs of read/write operations.

4.2 Intervals

The set of mathematical intervals is denoted by $\overline{\mathbb{IR}}$. It consists of exactly those subsets $\boldsymbol{x}$ of the real line $\mathbb{R}$ that are closed and connected in the topological sense. Thus, it comprises the empty set (denoted $\emptyset$ or Empty) together with all the nonempty intervals, denoted $[\underline{x}, \overline{x}]$ and defined by

$$[\underline{x}, \overline{x}] = \{\, x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x} \,\}, \tag{1}$$

where $\underline{x}$ and $\overline{x}$, the **bounds** of the interval, are extended-real numbers satisfying $\underline{x} \le \overline{x}$, $\underline{x} < +\infty$ and $\overline{x} > -\infty$.

This definition implies $-\infty$ and $+\infty$ can be bounds of an interval, but are never members of it. In particular, $[-\infty, +\infty]$ is the set of all *real* numbers satisfying $-\infty \le x \le +\infty$, which is the whole real line $\mathbb{R}$—not the whole extended real line $\overline{\mathbb{R}}$. Another name for the whole real line is Entire.

NOTE 1—The set of intervals $\overline{\mathbb{IR}}$ could be described more concisely as comprising all sets $\{\, x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x} \,\}$ for *arbitrary* extended-real $\underline{x}$, $\overline{x}$. However, this obtains Empty in many ways, as $[\underline{x}, \overline{x}]$ for any bounds satisfying $\underline{x} > \overline{x}$, and also as $[-\infty, -\infty]$ or $[+\infty, +\infty]$. The description (1) was preferred as it makes a one-to-one mapping between valid pairs $\underline{x}, \overline{x}$ of bounds and the nonempty intervals they specify.

A **box** or **interval vector** is an n-tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, whose components $\boldsymbol{x}_i \in \overline{\mathbb{IR}}$. The box $\boldsymbol{x}$ is empty if (and only if) any of its components $\boldsymbol{x}_i$ is empty.

4.3 Hull

The (interval) **hull** of an arbitrary subset $\boldsymbol{s}$ of $\mathbb{R}^n$, written $\text{hull}(\boldsymbol{s})$, is the tightest member of $\overline{\mathbb{IR}}^n$ that contains $\boldsymbol{s}$. Here the **tightest** set with a given property is the intersection of all sets having that property, provided the intersection itself has this property.

4.4 Functions

4.4.1 Function terminology

The terms operation, function and mapping are broadly synonymous. The following summarizes usage, with references in parentheses to precise definitions of terms.

- A *point function* (4.4.2) is a partial mathematical real function of real variables. Otherwise, *function* is usually used with its general mathematical meaning.
- An *arithmetic operation* (4.4.3) is a point function for which an implementation provides versions in the implementation's *library* (4.4.3).
- A *version* of a point function f means a function derived from f; typically a bare or decorated interval extension (4.4.4) of f.
- An *interval arithmetic operation* is an interval extension of a point arithmetic operation (4.4.4).
- An *interval non-arithmetic operation* is an interval-to-interval library function that is not an interval arithmetic operation (4.4.4).
- A *constructor* is a function that creates an interval from non-interval data (4.5.5).

4.4.2 Point function

A **point function** is a (possibly partial) multivariate real function: that is, a mapping f from a subset D of $\mathbb{R}^n$ to $\mathbb{R}^m$ for some integers $n \geq 0, m > 0$. It is a *scalar* function if $m = 1$, otherwise a *vector* function. When not otherwise specified, scalar is assumed.

The set D where f is defined is its **domain**, also written $\text{Dom}\, f$. To specify n, call f an n-variable point function, or denote values of f as

$$f(x_1, \dots, x_n).$$

The **range** of f over an arbitrary subset $\boldsymbol{s}$ of $\mathbb{R}^n$ is the set $\text{Rge}(f \mid \boldsymbol{s})$ defined by

$$\text{Rge}(f \mid \boldsymbol{s}) = \{\, f(x) \mid x \in \boldsymbol{s} \text{ and } x \in \text{Dom}\, f \,\}.$$

Thus mathematically, when evaluating a function over a set, points outside the domain are ignored—e.g., $\text{Rge}(\text{sqrt} \mid [-1, 1]) = [0, 1]$.

Equivalently, for the case where f takes separate arguments $\boldsymbol{s}_1, \dots, \boldsymbol{s}_n$, each being a subset of $\mathbb{R}$, the range is written as $\text{Rge}(f \mid \boldsymbol{s}_1, \dots, \boldsymbol{s}_n)$.

4.4.3 Point arithmetic operation

A (point) **arithmetic operation** is a function for which an implementation provides versions in a collection of user-available operations called its **library**. This includes functions normally written in operator form (e.g., $+$, $\times$) and those normally written in function form (e.g., exp, arctan). It is not specified (at Level 1) how an implementation provides library facilities.

4.4.4 Interval-valued functions

Let f be an n-variable scalar point function. An **interval extension** of f is a (total) mapping $\boldsymbol{f}$ from n-dimensional boxes to intervals, that is $\boldsymbol{f} : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$, such that $f(x) \in \boldsymbol{f}(\boldsymbol{x})$ whenever $x \in \boldsymbol{x}$ and $f(x)$ is defined, equivalently

$$\boldsymbol{f}(\boldsymbol{x}) \supseteq \operatorname{Rge}(f \mid \boldsymbol{x})$$

for any box $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$, regarded as a subset of $\mathbb{R}^n$.

The **natural interval extension** of f is the mapping $\boldsymbol{f}$ defined by

$$\boldsymbol{f}(\boldsymbol{x}) = \operatorname{hull}\big(\operatorname{Rge}(f \mid \boldsymbol{x})\big).$$

Equivalently, using multiple-argument notation for f, an interval extension satisfies

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \supseteq \operatorname{Rge}(f \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),$$

and the natural interval extension is defined by

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \operatorname{hull}\big(\operatorname{Rge}(f \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\big)$$

for any intervals $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

When f is a binary operator $\bullet$ written in infix notation, this gives the usual definition of its natural interval extension as

$$\boldsymbol{x} \bullet \boldsymbol{y} = \operatorname{hull}(\{\, x \bullet y \mid x \in \boldsymbol{x},\, y \in \boldsymbol{y}, \text{ and } x \bullet y \text{ is defined} \,\}).$$

[Example. With these definitions, the relevant natural interval extensions satisfy $\sqrt{[-1,4]} = [0,2]$ and $\sqrt{[-2,-1]} = \emptyset$; also $\boldsymbol{x} \times [0,0] = [0,0]$ for any nonempty $\boldsymbol{x}$, and $\boldsymbol{x}/[0,0] = \emptyset$, for any $\boldsymbol{x}$.]

When f is a vector point function, a vector interval function with the same number of inputs and outputs as f is called an interval extension of f, if each of its components is an interval extension of the corresponding component of f.

An interval-valued function in the library is called an

- **interval arithmetic operation**, if it is an interval extension of a point arithmetic operation, and an
- **interval non-arithmetic operation** otherwise.

Examples of the latter are interval intersection and convex hull, $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \cap \boldsymbol{y}$ and $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \operatorname{hull}(\boldsymbol{x} \cup \boldsymbol{y})$.

4.4.5 Constants

A real scalar function with no arguments—a mapping $\mathbb{R}^n \to \mathbb{R}^m$ with $n = 0$ and $m = 1$—is a **real constant**. An interval extension of a real constant is any zero-argument interval function that returns an interval containing c. The *natural extension* returns the interval $[c, c]$.

4.5 Required operations

4.5.1 Interval constants

The constant functions `empty`() and `entire`() have value Empty and Entire respectively.

4.5.2 Arithmetic operations

Table 4.1 lists required arithmetic operations, including those normally written in function notation $f(x, y, \ldots)$ and those normally written in unary or binary operator notation, $\bullet x$ or $x \bullet y$.

Each one is continuous at each point of its domain, except where stated in the footnotes after this table. Square and round brackets are used to include or exclude an interval bound, e.g., $(-\pi, \pi]$ denotes $\{\, x \in \mathbb{R} \mid -\pi < x \leq \pi \,\}$.

Table 4.1. Required forward elementary functions.

Name	Definition	Point function domain	Point function range	Table Footnotes
Basic operations				
neg(x)	$-x$	$\mathbb{R}$	$\mathbb{R}$	
add(x, y)	$x + y$	$\mathbb{R}^2$	$\mathbb{R}$	
sub(x, y)	$x - y$	$\mathbb{R}^2$	$\mathbb{R}$	
mul(x, y)	xy	$\mathbb{R}^2$	$\mathbb{R}$	
div(x, y)	x/y	$\mathbb{R}^2 \setminus \{y = 0\}$	$\mathbb{R}$	a
recip(x)	$1/x$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$	
sqr(x)	x^2	$\mathbb{R}$	$[0, \infty)$	
sqrt(x)	$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$	
fma(x, y, z)	$(x \times y) + z$	$\mathbb{R}^3$	$\mathbb{R}$	
Power functions				
pown(x, p)	$x^p,\ p \in \mathbb{Z}$	$\begin{cases} \mathbb{R} \text{ if } p \geq 0 \\ \mathbb{R}\setminus\{0\} \text{ if } p < 0 \end{cases}$	$\begin{cases} \mathbb{R} \text{ if } p > 0 \text{ odd} \\ [0, \infty) \text{ if } p > 0 \text{ even} \\ \{1\} \text{ if } p = 0 \\ \mathbb{R}\setminus\{0\} \text{ if } p < 0 \text{ odd} \\ (0, \infty) \text{ if } p < 0 \text{ even} \end{cases}$	b
pow(x, y)	x^y	$\{x{>}0\} \cup \{x{=}0, y{>}0\}$	$[0, \infty)$	a, c
exp,exp2,exp10(x)	b^x	$\mathbb{R}$	$(0, \infty)$	d
log,log2,log10(x)	$\log_b x$	$(0, \infty)$	$\mathbb{R}$	d
Trigonometric/hyperbolic				
sin(x)		$\mathbb{R}$	$[-1, 1]$	
cos(x)		$\mathbb{R}$	$[-1, 1]$	
tan(x)		$\mathbb{R}\setminus\{(k + \frac{1}{2})\pi \mid k \in \mathbb{Z}\}$	$\mathbb{R}$	
asin(x)		$[-1, 1]$	$[-\pi/2, \pi/2]$	e
acos(x)		$[-1, 1]$	$[0, \pi]$	e
atan(x)		$\mathbb{R}$	$(-\pi/2, \pi/2)$	e
atan2(y, x)		$\mathbb{R}^2 \setminus \{\langle 0, 0\rangle\}$	$(-\pi, \pi]$	e, f, g
sinh(x)		$\mathbb{R}$	$\mathbb{R}$	
cosh(x)		$\mathbb{R}$	$[1, \infty)$	
tanh(x)		$\mathbb{R}$	$(-1, 1)$	
asinh(x)		$\mathbb{R}$	$\mathbb{R}$	
acosh(x)		$[1, \infty)$	$[0, \infty)$	
atanh(x)		$(-1, 1)$	$\mathbb{R}$	
Integer functions				
sign(x)		$\mathbb{R}$	$\{-1, 0, 1\}$	h
ceil(x)		$\mathbb{R}$	$\mathbb{Z}$	i
floor(x)		$\mathbb{R}$	$\mathbb{Z}$	i
trunc(x)		$\mathbb{R}$	$\mathbb{Z}$	i
roundTiesToEven(x)		$\mathbb{R}$	$\mathbb{Z}$	j
roundTiesToAway(x)		$\mathbb{R}$	$\mathbb{Z}$	j
Absmax functions				
abs(x)	$\lvert x \rvert$	$\mathbb{R}$	$[0, \infty)$	
min(x, y)		$\mathbb{R}^2$	$\mathbb{R}$	k
max(x, y)		$\mathbb{R}^2$	$\mathbb{R}$	k

Footnotes to Table 4.1

a. In describing the domain, notation such as $\{y = 0\}$ is short for $\{ (x, y) \in \mathbb{R}^2 \mid y = 0 \}$, etc.

b. Regarded as a family of functions of one real variable x, parameterized by the integer argument p.

c. Defined as $e^{y \ln x}$ for real $x > 0$ and all real y, and 0 for $x = 0$ and $y > 0$, else has no value. It is continuous at each point of its domain, including the positive y axis which is on the boundary of the domain.

d. $b = e, 2$ or 10, respectively.

e. The ranges shown are the mathematical range of the point function. To ensure containment, an interval result may include values outside the mathematical range.

f. atan2(y, x) is the principal value of the argument (polar angle) of (x, y) in the plane. It is discontinuous on the half-line $y = 0, x < 0$ contained within its domain.

g. To avoid confusion with notation for open intervals, in this table coordinates in $\mathbb{R}^2$ are delimited by angle brackets $\langle\ \rangle$.

h. `sign`(x) is -1 if $x < 0$; 0 if $x = 0$; and 1 if $x > 0$. It is discontinuous at 0 in its domain.

i. `ceil`(x) is the smallest integer $\geq x$. `floor`(x) is the largest integer $\leq x$. `trunc`(x) is the nearest integer to x in the direction of zero. `ceil` and `floor` are discontinuous at each integer. `trunc` is discontinuous at each nonzero integer.

j. `roundTiesToEven`(x), `roundTiesToAway`(x) are the nearest integer to x, with ties rounded to the even integer or away from zero, respectively. They are discontinuous at each $x = n + \frac{1}{2}$ where n is an integer.

k. Smallest, or largest, of its real arguments.

4.5.3 Cancellative addition and subtraction

For intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, cancellative subtraction, `cancelMinus`$(\boldsymbol{x}, \boldsymbol{y})$, determines the tightest interval $\boldsymbol{z}$ such that

$$\boldsymbol{y} + \boldsymbol{z} \supseteq \boldsymbol{x},$$

if such a $\boldsymbol{z}$ exists.

At Level 1,

$$\boldsymbol{z} = \begin{cases} \emptyset & \text{if } \boldsymbol{x} = \emptyset \text{ and } \boldsymbol{y} \text{ is bounded} \\ [\underline{x} - \underline{y}, \overline{x} - \overline{y}] & \text{if } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are both nonempty and bounded and } \overline{y} - \underline{y} \leq \overline{x} - \underline{x}, \end{cases} \tag{2}$$

and $\boldsymbol{z}$ has no value in the cases when

$$\begin{cases} \text{either of } \boldsymbol{x} \text{ or } \boldsymbol{y} \text{ is unbounded, or} \\ \boldsymbol{x} \neq \emptyset \text{ and } y = \emptyset, \text{ or,} \\ \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are both nonempty and bounded and } \overline{y} - \underline{y} > \overline{x} - \underline{x}. \end{cases} \tag{3}$$

The operation `cancelPlus`$(\boldsymbol{x}, \boldsymbol{y})$ is equivalent to `cancelMinus`$(\boldsymbol{x}, -\boldsymbol{y})$, and therefore not considered separately.

4.5.4 Set operations The intersection and convex hull operations shall be provided as in Table 4.2.

Table 4.2. Set operations.

Name	Value
`intersection`$(\boldsymbol{a}, \boldsymbol{b})$	intersection $\boldsymbol{a} \cap \boldsymbol{b}$ of the intervals $\boldsymbol{a}$ and $\boldsymbol{b}$
`convexHull`$(\boldsymbol{a}, \boldsymbol{b})$	interval hull of the union $\boldsymbol{a} \cup \boldsymbol{b}$ of the intervals $\boldsymbol{a}$ and $\boldsymbol{b}$

4.5.5 Constructors

An interval constructor is an operation that creates a bare or decorated interval from non-interval data. The constructors `numsToInterval` and `textToInterval` shall be provided with values as defined below:

$$\texttt{numsToInterval}(l, u) = \begin{cases} [l, u] = \{ x \in \mathbb{R} \mid l \leq x \leq u \} & \text{if } l \leq u,\ l < +\infty \text{ and } u > -\infty \\ \text{no value} & \text{otherwise,} \end{cases}$$

where l and u are extended-real values; and

$$\texttt{textToInterval}(\boldsymbol{s}) = \begin{cases} \text{interval denoted by } \boldsymbol{s} & \text{if } \boldsymbol{s} \text{ is a valid interval literal (see 6.6)} \\ \text{no value} & \text{otherwise.} \end{cases}$$

4.5.6 Numeric functions of intervals

The operations in Table 4.3 shall be provided, the argument being an interval and the result a number, which for some of the operations may be infinite.

Implementations should provide an operation that returns `mid`($\boldsymbol{x}$) and `rad`($\boldsymbol{x}$) simultaneously.

Table 4.3. Required numeric functions of an interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$.
Note `inf` can have value $-\infty$; each of `sup`, `wid`, `rad` and `mag` can have value $+\infty$.

Name	Definition
$\inf(\boldsymbol{x})$	$\begin{cases} \text{lower bound of } \boldsymbol{x}, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ +\infty, \text{ if } \boldsymbol{x} \text{ is empty} \end{cases}$
$\sup(\boldsymbol{x})$	$\begin{cases} \text{upper bound of } \boldsymbol{x}, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ -\infty, \text{ if } \boldsymbol{x} \text{ is empty} \end{cases}$
$\text{mid}(\boldsymbol{x})$	$\begin{cases} \text{midpoint } (\underline{x} + \overline{x})/2, \text{ if } \boldsymbol{x} \text{ is nonempty bounded} \\ \text{no value, if } \boldsymbol{x} \text{ is empty or unbounded} \end{cases}$
$\text{wid}(\boldsymbol{x})$	$\begin{cases} \text{width } \overline{x} - \underline{x}, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ \text{no value, if } \boldsymbol{x} \text{ is empty} \end{cases}$
$\text{rad}(\boldsymbol{x})$	$\begin{cases} \text{radius } (\overline{x} - \underline{x})/2, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ \text{no value, if } \boldsymbol{x} \text{ is empty} \end{cases}$
$\text{mag}(\boldsymbol{x})$	$\begin{cases} \text{magnitude } \sup\{\, \lvert x \rvert \mid x \in \boldsymbol{x} \,\}, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ \text{no value, if } \boldsymbol{x} \text{ is empty} \end{cases}$
$\text{mig}(\boldsymbol{x})$	$\begin{cases} \text{mignitude } \inf\{\, \lvert x \rvert \mid x \in \boldsymbol{x} \,\}, \text{ if } \boldsymbol{x} \text{ is nonempty} \\ \text{no value, if } \boldsymbol{x} \text{ is empty} \end{cases}$

4.5.7 Boolean functions of intervals

The six boolean functions in Tables 4.4 and 4.5 shall be provided. In this document, the boolean values `true` and `false` are given numeric values 1 and 0 respectively, but a language may have a different mapping of booleans to numbers, or no such mapping.

Table 4.4. The `isEmpty` and `isEntire` functions.

Name	Returns
`isEmpty`($\boldsymbol{x}$)	1 if $\boldsymbol{x}$ is the empty set, 0 otherwise
`isEntire`($\boldsymbol{x}$)	1 if $\boldsymbol{x}$ is the whole line, 0 otherwise

In Table 4.5, column three gives the set-theoretic definition, and column four gives an equivalent specification when both intervals are nonempty. Table 4.6 shows what the definitions imply when at least one interval is empty.

Table 4.5. Comparisons for intervals a and b.
Notation $\forall_a$ means "for all a in $\boldsymbol{a}$", and so on. In column 4, $\boldsymbol{a}=[\underline{a},\overline{a}]$ and $\boldsymbol{b}=[\underline{b},\overline{b}]$, where $\underline{a},\underline{b}$ may be $-\infty$ and $\overline{a},\overline{b}$ may be $+\infty$.

Name	Symbol	Definition	For $\boldsymbol{a},\boldsymbol{b} \neq \emptyset$	Description
`equal`$(\boldsymbol{a},\boldsymbol{b})$	$\boldsymbol{a} = \boldsymbol{b}$	$\forall_a \exists_b\, a = b \wedge \forall_b \exists_a\, b = a$	$\underline{a} = \underline{b} \wedge \overline{a} = \overline{b}$	$\boldsymbol{a}$ equals $\boldsymbol{b}$
`subset`$(\boldsymbol{a},\boldsymbol{b})$	$\boldsymbol{a} \subseteq \boldsymbol{b}$	$\forall_a \exists_b\, a = b$	$\underline{b} \leq \underline{a} \wedge \overline{a} \leq \overline{b}$	$\boldsymbol{a}$ is a subset of $\boldsymbol{b}$
`interior`$(\boldsymbol{a},\boldsymbol{b})$	$\boldsymbol{a} \Subset \boldsymbol{b}$	$\forall_a \exists_b\, a < b \wedge \forall_a \exists_b\, b < a$	$\underline{b} < \underline{a} \wedge \overline{a} < \overline{b}$	$\boldsymbol{a}$ is interior to $\boldsymbol{b}$
`disjoint`$(\boldsymbol{a},\boldsymbol{b})$	$\boldsymbol{a} \mathbin{\not\cap} \boldsymbol{b}$	$\forall_a \forall_b\, a \neq b$	$\overline{a} < \underline{b} \vee \overline{b} < \underline{a}$	$\boldsymbol{a}$ and $\boldsymbol{b}$ are disjoint

Table 4.6. Comparisons with empty intervals.

	$\boldsymbol{a} = \emptyset$, $\boldsymbol{b} \neq \emptyset$	$\boldsymbol{a} \neq \emptyset$, $\boldsymbol{b} = \emptyset$	$\boldsymbol{a} = \emptyset$, $\boldsymbol{b} = \emptyset$
$\boldsymbol{a} = \boldsymbol{b}$	0	0	1
$\boldsymbol{a} \subseteq \boldsymbol{b}$	1	0	1
$\boldsymbol{a} \Subset \boldsymbol{b}$	1	0	1
$\boldsymbol{a} \mathbin{\not\cap} \boldsymbol{b}$	1	1	1