Example 1 *In signal processing, a quadratic convolution kernel has the piecewise definition*

$$h(x) = \begin{cases} 3/4 - |x|^2 & \textit{if } |x| \le 1/2, \\ 1/2 \cdot |x|^2 - 3/2 \cdot |x| + 9/8 & \textit{if } 1/2 \le |x| \le 3/2, \\ 0 & \textit{otherwise.} \end{cases} \tag{1}$$

A valid interval extension $H(X)$ of $h(x)$ is

$$H(X) = U(X) \cup V(X) \cup W(X) \tag{2}$$

where

$$U(X) = 3/4 - (|X| \cap [0, 1/2])^2 , \tag{3}$$
$$V(X) = 1/2 \cdot (|X| \cap [1/2, 3/2])^2 - 3/2 \cdot (|X| \cap [1/2, 3/2]) + 9/8, \tag{4}$$
$$W(X) = 0 \cdot (|X| \cap [3/2, \infty]) . \tag{5}$$

What is the decorated interval result of $H(X)$ when $X = [1/5, 1]$?

Since we have $|X| = [1/5, 1]$, all of the intersections in (3) and (4) are nonempty, giving the decorated interval results

$$U([1/5, 1]) = ([1/2, 71/100], \mathbb{D}_3), \tag{6}$$
$$V([1/5, 1]) = ([-1/4, 7/8], \mathbb{D}_3). \tag{7}$$

For (5), we have the intersection

$$[1/5, 1] \cap [3/2, \infty] = \varnothing$$

between the two disjoint intervals. The intersection operation propagates the worst decoration of the input operands, in this case $\mathbb{D}_3$; so we have

$$W([1/5, 1]) = 0 \cdot (\varnothing, \mathbb{D}_3) . \tag{8}$$

So far so good, i. e., both of our motions [1] and [2] are in agreement. But now we have in (8) an arithmetic operation where one of the input operands is empty, and by Definition 9 in [2] we have

$$\begin{aligned} W([1/5, 1]) &= 0 \cdot (\varnothing, \mathbb{D}_3) , \\ &= (\varnothing, \mathbb{D}_0) . \end{aligned}$$

Therefore

$$H([1/5, 1]) = ([-1/4, 7/8], \mathbb{D}_0). \tag{9}$$

Note the computed decoration for $H(X)$ is "undefined," even though $h(x)$ is defined and continuous for all $x \in X$. This is clearly the wrong result!

On the other hand, by definitions in [1], we have

$$\begin{aligned}
W([1/5,1]) &= 0 \cdot (\varnothing, \mathbb{D}_3)\,, \\
&= (0, \mathbb{D}_3) \cdot (\varnothing, \mathbb{D}_3)\,, \\
&= (0 \cdot \varnothing, \inf(S(\cdot, 0, \varnothing), \mathbb{D}_3, \mathbb{D}_3)), \\
&= (\varnothing, \inf(\mathbb{D}_4, \mathbb{D}_3, \mathbb{D}_3)), \\
&= (\varnothing, \mathbb{D}_3).
\end{aligned}$$

Therefore

$$H([1/5,1]) = ([-1/4, 7/8], \mathbb{D}_3). \tag{10}$$

The computed decoration for $H(X)$ in this case is "defined and continuous," and the contradiction is repaired.

References

[1] Hayes, N., "Property Tracking with Decorations," P1788 Motion 25-A1, May 28, 2011.

[2] Nehmeier, M. and J. Wolff von Gudenberg, "Decorated Intervals A Pragmatic Approach," P1788 Motion 27, June 29, 2011.