4.4.4 Interval-valued functions

Let f be an n-variable scalar point function. An **interval extension** of f is a (total) mapping $\boldsymbol{f}$ from n-dimensional boxes to intervals, that is $\boldsymbol{f} : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$, such that $f(x) \in \boldsymbol{f}(\boldsymbol{x})$ whenever $x \in \boldsymbol{x}$ and $f(x)$ is defined, equivalently

$$\boldsymbol{f}(\boldsymbol{x}) \supseteq \mathrm{Rge}(f \mid \boldsymbol{x})$$

for any box $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$, regarded as a subset of $\mathbb{R}^n$.

The **natural interval extension** of f is the mapping $\boldsymbol{f}$ defined by

$$\boldsymbol{f}(\boldsymbol{x}) = \mathrm{hull}\big(\mathrm{Rge}(f \mid \boldsymbol{x})\big).$$

Equivalently, using multiple-argument notation for f, an interval extension satisfies

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \supseteq \mathrm{Rge}(f \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),$$

and the natural interval extension is defined by

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \mathrm{hull}\big(\mathrm{Rge}(f \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\big)$$

for any intervals $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

NOTE—The term "natural interval extension" has been used in the literature to denote an evaluation of a function in interval arithmetic, where scalars are replaced by intervals and point operations are replaced by interval ones. The result of such an evaluation generally depends on the particular expression for $\boldsymbol{f}$, while the present definition of "natural interval extension" specifies unambiguously a unique result.

When f is a binary operator $\bullet$ written in infix notation, this gives the usual definition of its natural interval extension as

$$\boldsymbol{x} \bullet \boldsymbol{y} = \mathrm{hull}(\{\, x \bullet y \mid x \in \boldsymbol{x},\, y \in \boldsymbol{y}, \text{ and } x \bullet y \text{ is defined} \,\}).$$

[Example. With these definitions, the relevant natural interval extensions satisfy $\sqrt{[-1,4]} = [0,2]$ and $\sqrt{[-2,-1]} = \emptyset$; also $\boldsymbol{x} \times [0,0] = [0,0]$ for any nonempty $\boldsymbol{x}$, and $\boldsymbol{x}/[0,0] = \emptyset$, for any $\boldsymbol{x}$.]

When f is a vector point function, a vector interval function with the same number of inputs and outputs as f is called an interval extension of f, if each of its components is an interval extension of the corresponding component of f.

An interval-valued function in the library is called an

– **interval arithmetic operation**, if it is an interval extension of a point arithmetic operation, and an

– **interval non-arithmetic operation** otherwise.

Examples of the latter are interval intersection and convex hull, $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \cap \boldsymbol{y}$ and $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathrm{hull}(\boldsymbol{x} \cup \boldsymbol{y})$.

4.4.5 Constants

A real scalar function with no arguments—a mapping $\mathbb{R}^n \to \mathbb{R}^m$ with $n = 0$ and $m = 1$—is a **real constant**. An interval extension of a real constant is any zero-argument interval function that returns an interval containing c. The *natural extension* returns the interval $[c, c]$.

4.5 Required operations

4.5.1 Interval constants

The constant functions `empty`() and `entire`() have value Empty and Entire respectively.