Motion 10v2.2: Elementary Functions

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1 Motion

This motion provides a list of elementary functions that should be required for P1788.

1.1 Related Motions

We assume that an accuracy model will be defined for the standard. The Vienna Proposal, e.g., specifies the modes "tightest", "accurate", and "valid". In this motion we do not specify the accuracy model.

Some of the functions will also be provided in "reverse mode", but that shall be subject of another motion.

Evaluation of functions outside their domain shall be handled with the common exception handling scheme. Currently that means decorated intervals proposed as motion 8 that is still under discussion.

1.2 List of Functions

P1788 shall provide a fixed set of mandatory interval elementary functions defined as interval extensions of the point elementary functions that are commonly used in scientific environments. In particular most of the functions proposed in IEEE 754 and in the Vienna proposal are considered. Table 1 lists the (P1788) name of the function, its mathematical definition with the rough domain (that means the domain including the points where the function has an isolated singularity) of the real function as well as the *domain* → *range* characterization of the interval function.

Given the real function f with domain D_f and range R_f

$f : D_f \to R_f, \quad x \mapsto f(x)$

the interval extension is defined as

$f : \mathbb{I}D_f \to \mathbb{I}R_f, \quad \mathbf{x} \mapsto \Diamond\{f(x) | x \in \mathbf{x} \cap D_f\}$

Here $\mathbb{I}M$ denotes the set of all intervals with endpoints in M. This definition is in accordance with motion 5 (arithmetic operators). Here $\Diamond$ means the interval hull. That means that $f(\mathbf{x}) \in \overline{\mathbb{IR}}$ on level 1 and $f : \overline{\mathbb{IF}} \to \overline{\mathbb{IF}}, \mathbf{x} \mapsto \Diamond f(\mathbf{x})$ on level 2. Level 3 tables are not part of the motion. A sample table, however, is given in the rationale.

1.3 General Power Function

Usually the interval power function x^y is defined as extension of the real function $e^{y \cdot \ln(x)}$. Hence, x has to be positive.

But actually $x^{p/q}$ with $x < 0$ is defined for rational exponents, if q is odd. If p is odd the result is negative otherwise positive. Since the rationals are dense in the reals, we can find a rational number with odd denominator and odd or even numerator in each neighborhood of a negative real number. Hence, we can define an interval version of the power function that directly delivers the interval between the 2 values.

The general interval power function called *pow* is defined as follows (Let $\underline{\cup}$ compute the interval hull of its 2 operands.)

$pow : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \rightarrow \overline{\mathbb{IR}}$.

If $\mathbf{x} > 0 : pow(\mathbf{x}, \mathbf{y}) = \exp(\mathbf{y} \cdot \ln(\mathbf{x}))$

If $\mathbf{x} < 0 : pow(\mathbf{x}, \mathbf{y}) = -\exp(\mathbf{y} \cdot \ln(|\mathbf{x})|) \underline{\cup} \exp(\mathbf{y} \cdot \ln(|\mathbf{x})|)$

If $0 \in \mathbf{x} : pow(\mathbf{x}, \mathbf{y}) = pow([\underline{x}, 0), \mathbf{y}) \underline{\cup} pow([0, 0], \mathbf{y}) \underline{\cup} pow((0, \overline{x}], \mathbf{y})$

where $pow([0, 0], \mathbf{y}) = [0, 0]$, if $\exists 0 < y \in \mathbf{y}$

name	definition	rough domain	interval function	note
sqr	x^2	$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}[0,\infty)$	1
pown	x^n	$\mathbb{R} \times \mathbb{Z}$	$\overline{\mathbb{IR}} \times \mathbb{Z} \to \overline{\mathbb{IR}}$	2, 3
pow	x^y		$\overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$	4
sqrt		$[0,\infty)$	$\mathbb{I}[0,\infty) \to \mathbb{I}[0,\infty)$	1
exp, exp2, exp10	e^x, b^x	$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}(0,\infty)$	5
log,log2,log10	$\ln x, \log_b x,$	$(0,\infty)$	$\mathbb{I}(0,\infty) \to \overline{\mathbb{IR}}$	5
expm1, exp2m1, exp10m1	$e^x - 1$, $b^x - 1$	$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}(-1,\infty)$	5
logp1, log2p1, log10p1	$\ln(x+1)$, $\log_b(x+1)$	$(-1,\infty)$	$\mathbb{I}(-1,\infty) \to \overline{\mathbb{IR}}$	5
sin		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}[-1,1]$	
cos		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}[-1,1]$	
tan		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$	6
asin		$[-1,1]$	$\mathbb{I}[-1,1] \to \mathbb{I}[-\pi/2,\pi/2]$	7
acos		$[-1,1]$	$\mathbb{I}[-1,1] \to \mathbb{I}[-\pi/2,\pi/2]$	7
atan		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}(-\pi/2,\pi/2)$	7
atan2		$\mathbb{R} \times \mathbb{R}$	$\overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \mathbb{I}(-\pi,\pi]$	7, 8
sinh		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$	
cosh		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}[1,\infty)$	
tanh		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}(-1,1)$	
asinh		$\mathbb{R}$	$\overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$	
acosh		$[1,\infty)$	$\mathbb{I}[1,\infty) \to \mathbb{I}[0,\infty)$	
atanh		$[-1,1]$	$\mathbb{I}[-1,1] \to \overline{\mathbb{IR}}$	
abs	$\lvert x \rvert$	$\mathbb{R}$	$\overline{\mathbb{IR}} \to \mathbb{I}[0,\infty)$	1
rSqrt	$1/\sqrt{x}$	$(0,\infty)$	$\mathbb{I}(0,\infty) \to \mathbb{I}(0,\infty)$	
hypot	$\sqrt{x^2+y^2}$	$\mathbb{R} \times \mathbb{R}$	$\overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \mathbb{I}[0,\infty)$	
compoundm1	$(1+x)^n - 1$	$(-1,\infty) \times \mathbb{Z}$	$\mathbb{I}(-1,\infty) \times \mathbb{Z} \to \mathbb{I}[-1,\infty)$	2, 9

Table 1: Required elementary functions

Notes:

1. Tightest, i.e. least bit accuracy
2. The integer argument selects the proper function out of a parameterized family.
3. For $n \leq 0$ the function is not defined for $x = 0$.
4. No extension of a real function, hence no rough domain (see 1.3).
5. $b = 2$ or $b = 10$, respectively
6. tan is not defined for $\{(2k+1)\pi/2 | k \in \mathbb{Z}\}$
7. The ranges shown in the table above are the mathematical range of the point function. The interval result must satisfy the containment rule and may thus include values just outside the mathematical range.

8. atan2(y,x) is the Principal Value of the argument of (x,y). atan2 is not defined for $y = 0, x = 0$

9. For $n \leq 0$ the function is not defined for $x = -1$.

1.4 Appendix: Further, recommended functions

More functions are listed in Table 2 which is not normative, it may be considered as an appendix.

The list is informative and not subject to vote.

name	definition	domain	range	notes
sign		$\mathbb{R}$	$\{-1, 0, 1\}$	
ceil, floor		$\mathbb{R}$	$\mathbb{Z}$	
nint, trunc		$\mathbb{R}$	$\mathbb{Z}$	
rootn	$x^{(1/q)}$	$\{(x, q) \in \mathbb{R} \times \mathbb{Z} \| q \neq 0,$ $x = 0 \Rightarrow q > 0, x < 0 \Rightarrow q \text{ odd}\}$	$\mathbb{R}$	2
powr	$x^{(p/q)}$	$\{(x, n, m) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \|$	$\mathbb{R}$	2
Let $p, q \in \mathbb{Z}$, such that $n/m = p/q, q > 0, gcd(p, q) = 1, x = 0 \Rightarrow p > 0, x < 0 \Rightarrow q \text{ odd}\}$				
sinPi	$\sin(\pi x)$	$\mathbb{R}$	$[-1, 1]$	
cosPi	$\cos(\pi x)$	$\mathbb{R}$	$[-1, 1]$	
atanPi	$\text{atan}(x)/\pi$	$\mathbb{R}$	$[-1/2, 1/2]$	
atan2Pi	$\text{atan2}(y, x)/\pi$	$\mathbb{R} \times \mathbb{R}$	$[-1, 1]$	
exp1x	$(e^x - 1)/x$	$\mathbb{R}$	$\mathbb{R}$	
exp2x	$(e^x - 1 - x)/x^2$	$\mathbb{R}$	$(0, \infty)$	
cos2	$(\cos(x) - 1)/x^2$	$\mathbb{R}$	$(-\infty, 0]$	
sin3	$3(\sin x - x)/x^3$	$\mathbb{R}$	$[-1/2, \infty)$	
cosh2	$(\cosh(x) - 1)/x^2$	$\mathbb{R}$	$(0, \infty)$	
sinh3	$3(\sinh x - x)/x^3$	$\mathbb{R}$	$[1/2, \infty)$	
gamma	$\Gamma(x)$	$\mathbb{R}$	$\mathbb{R}$	
lgamma	$\log \|\Gamma(x)\|$	$\mathbb{R}$	$\mathbb{R}$	
erf	error fct	$\mathbb{R}$	$(-1, 1)$	
erfc	compl error fct	$\mathbb{R}$	$(-1, 1)$	

Table 2: Recommended elementary functions