Practical Interval Arithmetic

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1. Interval Sets and Mappings

Interval arithmetic over the real numbers deals with closed and connected sets of the real numbers $\mathbb{R}$. An interval is denoted by an ordered pair. The first element is the lower bound and the second is the upper bound. The lower bound shall not be greater than the upper bound. If an interval is bounded it is written as $[a, b]$, with $a, b \in \mathbb{R}$. If it is unbounded it is written as $(-\infty, a]$ or $[b, +\infty)$ with $a, b \in \mathbb{R}$ or $(-\infty, +\infty)$ where the parentheses indicate that the bounds $-\infty$ and $+\infty$ are not elements of the interval. The set of all such intervals is denoted by $\mathbb{IR}$. With respect to set inclusion as an order relation $\{\mathbb{IR}, \subseteq\}$ is a complete lattice. It is bounded from below by the empty set $\varnothing$ and from above by the set $(-\infty, +\infty)$. If an interval is denoted by a single letter, the lower bound is denoted by a subscript 1 and the upper bound is denoted by the subscript 2.

Arithmetic for real numbers as well as for sets of real numbers is well defined. For interval operands the result of an operation always leads to an interval again and the bounds of the result can be expressed by simple expressions for the bounds of the operands. This gives interval arithmetic the right to exist. The corresponding formulas can be deduced from the definition of the operations for sets of real numbers in a strict mathematical manner [5, 6]. The calculus $\{\mathbb{IR}, +, -, *, /\}$ is free of exceptions.

On the computer real numbers are approximated by the subset of floating-point numbers as defined by the IEEE P754 floating-point arithmetic standard, for instance. The set of all floating-point numbers is denoted by $\mathbb{F}$. The subset of all bounded or unbounded intervals of $\mathbb{IR}$ with finite bounds of $\mathbb{F}$ is denoted by $\mathbb{IF}$. Intervals of $\mathbb{IR}$, arithmetic operations, and comparison relations for these are approximated by intervals, arithmetic operations, and comparison relations for intervals of the set $\mathbb{IF}$.

We consider here only the set of double precision binary or decimal floating-point numbers. For other floating-point formats and encodings the considerations are similar.

A real number or an interval over the real numbers is mapped onto the smallest floating-point interval that contains the number or interval respectively. This mapping $\diamondsuit : \mathbb{IR} \to \mathbb{IF}$ is characterized by the following properties:

(R1): $\diamondsuit \boldsymbol{a} = \boldsymbol{a}$, for all $\boldsymbol{a} \in \mathbb{IF}$,
(R2): $\boldsymbol{a} \subseteq \boldsymbol{b} \Rightarrow \diamondsuit \boldsymbol{a} \subseteq \diamondsuit \boldsymbol{b}$, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$,
(R3): $\boldsymbol{a} \subseteq \diamondsuit \boldsymbol{a}$, for all $\boldsymbol{a} \in \mathbb{IR}$,
(R4): $\diamondsuit (-\boldsymbol{a}) = - \diamondsuit \boldsymbol{a}$, for all $\boldsymbol{a} \in \mathbb{IR}$.

2. Arithmetic Operations for Intervals

The IEEE floating-point arithemtic standard P754 specifies arithmetic with four roundings: to the nearest floating-point number, downwards, upwards, and towards zero. For these operations the following notations will be used:

$+, -, *, /$ for the operations with rounding to the nearest floating-point number,
$\triangledown$, $\bigtriangledown$, $\overset{*}{\triangledown}$, $\bigtriangledown$ for the operations with rounding downwards,
$\triangle$, $\bigtriangleup$, $\overset{*}{\triangle}$, $\bigtriangleup$ for the operations with rounding upwards,[1] and
$*|, -|, +|, /|$ for the operations with rounding towards zero (chopping).[2]

With these notations for bounded intervals $a = [a_1, a_2], b = [b_1, b_2] \in \mathbb{IF}$ the following arithmetic operations $+, -, *$, and $/$ are defined:

Addition $[a_1, a_2] + [b_1, b_2] = [a_1 \triangledown b_1, a_2 \triangle b_2]$.

Subtraction $[a_1, a_2] - [b_1, b_2] = [a_1 \bigtriangledown b_2, a_2 \bigtriangleup b_1]$.

Multiplication $[a_1, a_2] * [b_1, b_2]$	$[b_1, b_2]$ $b_2 \leq 0$	$[b_1, b_2]$ $b_1 < 0 < b_2$	$[b_1, b_2]$ $b_1 \geq 0$
$[a_1, a_2], a_2 \leq 0$	$[a_2 \overset{*}{\triangledown} b_2, a_1 \overset{*}{\triangle} b_1]$	$[a_1 \overset{*}{\triangledown} b_2, a_1 \overset{*}{\triangle} b_1]$	$[a_1 \overset{*}{\triangledown} b_2, a_2 \overset{*}{\triangle} b_1]$
$a_1 < 0 < a_2$	$[a_2 \overset{*}{\triangledown} b_1, a_1 \overset{*}{\triangle} b_1]$	$[min(a_1 \overset{*}{\triangledown} b_2, a_2 \overset{*}{\triangledown} b_1),$ $max(a_1 \overset{*}{\triangle} b_1, a_2 \overset{*}{\triangle} b_2)]$	$[a_1 \overset{*}{\triangledown} b_2, a_2 \overset{*}{\triangle} b_2]$
$[a_1, a_2], a_1 \geq 0$	$[a_2 \overset{*}{\triangledown} b_1, a_1 \overset{*}{\triangle} b_2]$	$[a_2 \overset{*}{\triangledown} b_1, a_2 \overset{*}{\triangle} b_2]$	$[a_1 \overset{*}{\triangledown} b_1, a_2 \overset{*}{\triangle} b_2]$

Division, $0 \notin b$ $[a_1, a_2]/[b_1, b_2]$	$[b_1, b_2]$ $b_2 < 0$	$[b_1, b_2]$ $b_1 > 0$
$[a_1, a_2], a_2 \leq 0$	$[a_2 \bigtriangledown b_1, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, a_2 \bigtriangleup b_2]$
$[a_1, a_2], a_1 < 0 < a_2$	$[a_2 \bigtriangledown b_2, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, a_2 \bigtriangleup b_1]$
$[a_1, a_2], a_1 \geq 0$	$[a_2 \bigtriangledown b_2, a_1 \bigtriangleup b_1]$	$[a_1 \bigtriangledown b_2, a_2 \bigtriangleup b_1]$

Division, $0 \in b$ $[a_1, a_2]/[b_1, b_2]$	$b =$ $[0, 0]$	$[b_1, b_2]$ $b_1 < b_2 = 0$	$[b_1, b_2]$ $0 = b_1 < b_2$
$[a_1, a_2], a_2 < 0$	$\varnothing$	$[a_2 \bigtriangledown b_1, +\infty)$	$(-\infty, a_2 \bigtriangleup b_2]$
$[a_1, a_2], a_1 \leq 0 \leq a_2$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, a_2], a_1 > 0$	$\varnothing$	$(-\infty, a_1 \bigtriangleup b_1]$	$[a_1 \bigtriangledown b_2, +\infty)$

Division by an interval that includes zero in the last table leads to unbounded intervals. To be complete, arithmetic operations for unbounded intervals now have to be defined also.

[1] In our Pascal extension (available since 1980) and the Fortran extension we developed for and with IBM (available 1990) pairs of keybord symbols $+<, -<, *<, /<$ and $+>, ->, *>, />$ have been used for the operations with rounding downwads and upwards, respectively.

[2] Frequently used programming languages do not allow four plus, minus, multiply, and divide operators for floating-point numbers. A future interval arithmetic standard could or should specify names for low level operations with the directed roundings. They could be: $addp, subp, mulp, divp, addn, subn, muln$, and $divn$. Here p stands for rounding toward positive and n for rounding toward negative. With these operations interval routines would be fully transferable from one processor to another.

The first rule is that any operation with the empty set $\varnothing$ has the empty set as its result. Arithmetic operations for unbounded intervals of $\mathbb{IF}$ can be performed on the computer by using the above formulas for bounded intervals if in addition a few formal rules for operations with $-\infty$ and $+\infty$ are applied. These rules are shown in the following tables.

Addition	$-\infty$	b	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-$
a	$-\infty$	$-$	$+\infty$
$+\infty$	$-$	$+\infty$	$+\infty$

Subtraction	$-\infty$	b	$+\infty$
$-\infty$	$-$	$-\infty$	$-\infty$
a	$+\infty$	$-$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$-$

Multiplication	$-\infty$	$b<0$	0	$b>0$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	0	$-\infty$	$-\infty$
$a<0$	$+\infty$	$-$	$-$	$-$	$-\infty$
0	0	$-$	$-$	$-$	0
$a>0$	$-\infty$	$-$	$-$	$-$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	0	$+\infty$	$+\infty$

Division	$-\infty$	$+\infty$
a	0	0

These rules are not new in principle. They are well established in real analysis and IEEE P754 provides them anyway. The only rule that goes beyond IEEE P754 is

$$0 * (-\infty) = (-\infty) * 0 = 0 * (+\infty) = (+\infty) * 0 = 0. \tag{1}$$

This rule follows quite naturally from the definition of unbounded intervals. However, it should not be taken as a new mathematical law. It is just a short cut to easily compute the bounds of the result of an operation on unbounded intervals.

With the mapping $\diamondsuit : \mathbb{IR} \to \mathbb{IF}$ and its properties listed at the end of Section 1 the operations defined in this section have the following property which defines them uniquely:

(RG): $\boldsymbol{a} \diamondsuit \boldsymbol{b} := \diamondsuit (\boldsymbol{a} \circ \boldsymbol{b})$, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IF}$ and all $\circ \in \{+, -, *, /\}$.

3. Remarks on the Arithmetic Operations

I. In the table for division by an interval that includes zero the case $b_1 < 0 < b_2$ is missing. This needs some explanation.

A basic concept of mathematics is that of a function or mapping. A function consists of a pair (f, D_f). It maps each element x of its domain of definition D_f on a unique element y of the range R_f of f, $f : D_f \to R_f$.

In real analysis division by zero is not defined. Thus a rational function $y = f(x)$ where the denominator is zero for $x = c$ is not defined for $x = c$, i.e., c is not an element of the domain of definition D_f. Since the function $f(x)$ is not defined at $x = c$ it does not have any value or property there. In this strict mathematical sense, division by an interval $[b_1, b_2]$ with $b_1 < 0 < b_2$ is not well posed. For division the set $b_1 < 0 < b_2$ devolves into the two distinct sets $[b_1, 0]$[3] and $[0, b_2]$ and division by an interval $[b_1, b_2]$ with $b_1 < 0 < b_2$ actually consists of two divisions the result of which again consists

[3]Since division by zero does not contribute to the solution set it does not matter whether a paranthesis or bracket is used here.

of two distinct sets. In each case the result is a single unbounded interval. The two divisions should be performed separately. Division by the two sets $[b_1, 0]$ and $[0, b_2]$ is shown in the relevant table.

The situation is plainly shown by the signs of the bounds of the divisor before the division is executed. For interval multiplication or division a case selection has to be done (by hardware or software) anyhow before the operations are performed. In the case $b_1 < 0 < b_2$ the sign of b_1 is negative and the sign of b_2 is positive.

In the user's program, however, the two divisions appear within a single operation, as division by an interval $[b_1, b_2]$ with $b_1 < 0 < b_2$. So an arithmetic operation in the user's program delivers two distinct results. This is an unusual situation in conventional computing.[4]

A solution to the problem would be for the computer to provide a flag for *distinct intervals*. The situation occurs if the divisor is an interval that contains zero as an interior point. In this case the flag would be raised and signaled to the user. The user may then apply a routine of his choice to deal with the situation as is appropriate for his application.

This routine could be: Modify the operands and recompute, or continue the computation with one of the sets and ignore the other one, or put one of the sets on a list and continue the computation with the other one, or return the entire set of real numbers $(-\infty, +\infty)$ as result and continue the computation, or stop computing, or any other action.

A somewhat natural solution would be to continue the computation on different tasks, one for each interval. But the situation can occur repeatedly. How many tasks would we need? Future multicore processors will provide a large number of units and perhaps allow to run many tasks in parallel. A similar situation occurs in global optimization using subdivision. After a certain test several candidates may be left for further investigation.

Newton's method reaches its ultimate elegance and strength in the extended interval Newton method. It computes all (single) zeros in a given domain. If a function has several zeros in a given interval its derivative becomes zero in that interval also. Thus Newton's method applied to that interval delivers two distinct sets. This is how the extended interval Newton method separates different zeros. If the method is continued along two separate paths, one for each of the distinct intervals it finally computes all zeros in the given domain. If the method continues with only one of the two distinct sets and ignores the other one it computes an enclosure of only one zero of the given function. If the interval Newton method delivers the empty set, the method has proved that there is no zero in the initial interval.

II. If interval arithmetic is hardware supported then execution of the operations listed above is about as fast as execution of the corresponding floating-point operations. It is thus not reasonable to define and study operations between floating-point numbers and intervals in order to save computing time. Floating-point arithmetic and interval

[4]It would be very convenient for computing if other operations would also deliver two answers: floating-point addition and subtraction the rounded result and the error, multiplication the product to the double length and division the quotient and the remainder.

arithmetic are not the same calculus for approximate arithmetic for real numbers. They should be kept strictly separate.[5]

Of course, computing with result verification often makes use of floating-point computations. If executed in IEEE P754 arithmetic this may lead to exceptional results. So there remains the question of how results like $-\infty$, $+\infty$, NaN, -0, $+0$ can reasonably be mapped on floating-point intervals.

The following would be reasonable: -0 and $+0$ can only mean 0. Since NaN is not a real number it should be mapped on the empty set and since $-\infty$ and $+\infty$ are also not real numbers their image could or should also be the empty set. If the image of the result of a floating-point computation is the empty set the user should be informed.

III. The empty set $\varnothing$ may occur as result of an interval operation as listed in the tables of Section 2. The result of any operation with the empty set $\varnothing$ was defined to be the empty set. This suggests an encoding of the empty set by $\varnothing = [+NaN, -NaN]$. Then the rules for interval arithmetic listed in Section 2 can also be applied to the empty set. By the well established rules of IEEE P754 for NaN an operation with the empty set would then automatically produce the empty set as the result.

The encoding $\varnothing = [+NaN, -NaN]$ for the empty set also turns out to be useful for the definition of comparison relations for intervals. These will be studied in Section 5.

4. Variable Precision Interval Arithmetic

Success of interval arithmetic is based on two arithmetical features: One is double precision interval arithmetic. The other is variable precision interval arithmetic [1, 9, 10, 12]. For interval evaluation of an algorithm (a sequence of arithmetic operations) in the real number field a theorem by R. E. Moore [10] states that increasing the precision by k digits reduces the error bounds by b^{-k}, i.e., results can always be guaranteed to a number of correct digits by using variable precision interval arithmetic (for details see [1], [12]). Variable length interval arithmetic can be made very fast by an exact dot product and complete arithmetic [5], [8]. An exact dot product for the double precision format is the basic tool to achieve high speed variable (dynamic) precision arithmetic for real and interval data. Pipelining gives it high speed, and exactitude brings very high accuracy into computation. There is no way to compute a dot product faster than the exact method. By pipelining, it can be computed in the time the processor needs to read the data, i.e., it comes with utmost speed [4, 5]. Variable length interval arithmetic fully benefits from such speed [5]. No software simulation can go as fast. With operator overloading variable length interval arithmetic is very easy to use.

5. Comparison Relations and Lattice Operations

Three comparison relations are important for intervals of $\mathbb{IF}$:

(2) *equality*, *less than or equal*, and *set inclusion*.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be intervals of $\mathbb{IF}$ with bounds $a_1 \le a_2$ and $b_1 \le b_2$ respectively. Then the relations *equality* and *less than or equal* in $\mathbb{IF}$ are defined by:

$$\begin{aligned} \boldsymbol{a} = \boldsymbol{b} &:\Leftrightarrow a_1 = b_1 \wedge a_2 = b_2, \\ \boldsymbol{a} \le \boldsymbol{b} &:\Leftrightarrow a_1 \le b_1 \wedge a_2 \le b_2. \end{aligned}$$

[5]The XSC-languages allow real and interval data and operations between these in an expression. However, all real data are immediately interpreted as intervals and all operations are performed as interval operations.

Since bounds for intervals of $\mathbb{IF}$ may be $-\infty$ or $+\infty$ these comparison relations are executed as if performed in the lattice $\{\mathbb{F}^*, \leq\}$ with $\mathbb{F}^* := \mathbb{F} \cup \{-\infty\} \cup \{+\infty\}$.

With the order relation $\leq$, $\{\mathbb{IF}, \leq\}$ is a lattice. The *greatest lower bound* (glb) and the *least upper bound* (lub) of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IF}$ are the intervals

$$\begin{aligned} glb(\boldsymbol{a}, \boldsymbol{b}) &:= [min(a_1, b_1), min(a_2, b_2)], \\ lub(\boldsymbol{a}, \boldsymbol{b}) &:= [max(a_1, b_1), max(a_2, b_2)]. \end{aligned}$$

The greatest lower bound and the least upper bound of an interval with the empty set are both the empty set.

The inclusion relation in $\mathbb{IF}$ is defined by

$$\boldsymbol{a} \subseteq \boldsymbol{b} :\Leftrightarrow b_1 \leq a_1 \wedge a_2 \leq b_2. \tag{3}$$

With the relation $\subseteq$, $\{\mathbb{IF}, \subseteq\}$ is also a lattice. The least element in $\{\mathbb{IF}, \subseteq\}$ is the empty set $\varnothing$ and the greatest element is the interval $(-\infty, +\infty)$. The infimum of two elements $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IF}$ is the intersection and the supremum is the interval hull (convex hull):

$$\begin{aligned} inf(\boldsymbol{a}, \boldsymbol{b}) = \quad \boldsymbol{a} \cap \boldsymbol{b} \quad &:= [max(a_1, b_1), min(a_2, b_2)] \quad \text{or the empty set } \varnothing, \\ sup(\boldsymbol{a}, \boldsymbol{b}) = \quad \boldsymbol{a} \cup \boldsymbol{b} \quad &:= [min(a_1, b_1), max(a_2, b_2)]. \end{aligned}$$

The intersection of an interval with the empty set is the empty set. The interval hull with the empty set is the other operand.

If in the formulas for $glb(\boldsymbol{a}, \boldsymbol{b})$, $lub(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{a} \cap \boldsymbol{b}$, $\boldsymbol{a} \cup \boldsymbol{b}$, a bound is $-\infty$ or $+\infty$ a parenthesis should be used for this interval bound to denote the resulting interval. This bound is not an element of the interval.

If in any of the comparison relations defined here both operands are the empty set, the result is true. If in (3) $\boldsymbol{a}$ is the empty set the result is true. Otherwise the result is false if in any of the three comparison relations only one operand is the empty set.[6]

A particular case of inclusion is the relation *element of*. It is defined by

$$a \in \boldsymbol{b} :\Leftrightarrow b_1 \leq a \wedge a \leq b_2. \tag{4}$$

Another useful check is for whether $[a_1, a_2]$ is an interval at all, that is, if $a_1 \leq a_2$.

6. Evaluation of Functions

Let f be a function and D_f its domain of definition. For an interval $\boldsymbol{x} \subseteq D_f$, the range $range(f, \boldsymbol{x})$ of f is defined as the set of the function's values for all $x \in \boldsymbol{x}$:

$$range(f, \boldsymbol{x}) := \{f(x) | x \in \boldsymbol{x}\}. \tag{5}$$

On the computer, interval evaluation of a real function $f(x)$ for $\boldsymbol{x} \subseteq D_f$ should deliver a highly accurate enclosure of the range $range(f, \boldsymbol{x})$ of the function.

Evaluation of a function $f(x)$ for an interval $\boldsymbol{x}$ with $\boldsymbol{x} \cap D_f = \varnothing$, of course, does not make sense, since $f(x)$ is not defined for values outside its domain D_f. The empty set $\varnothing$ should be delivered and an error message may be given to the user.

[6] A convenient encoding of the empty set may be $\varnothing = [+NaN, -NaN]$. Then most comparison relations and lattice operations considered in this section would deliver the correct answer if conventional rules for NaN are applied. However, if $\boldsymbol{a} = \varnothing$ then set inclusion (3) and computing the interval hull do not follow this rule. So in these two cases whether $\boldsymbol{a} = \varnothing$ must be checked before the operations can be executed.

There are, however, applications in interval arithmetic where information about a function f is useful when $\boldsymbol{x}$ exceeds the domain D_f of f. The interval $\boldsymbol{x}$ may also be the result of overestimation during an earlier interval computation.

In such cases the range of f can only be computed for the intersection $\boldsymbol{x}' := \boldsymbol{x} \cap D_f$:

$$range(f, \boldsymbol{x}') := range(f, \boldsymbol{x} \cap D_f) := \{f(x) | x \in \boldsymbol{x} \cap D_f\}.$$

To prevent the wrong conclusions being drawn, the user must be informed that the interval $\boldsymbol{x}$ had to be reduced to $\boldsymbol{x}' := \boldsymbol{x} \cap D_f$ to compute the delivered range. A particular flag for *domain overflow* may serve this purpose. An appropriate routine can be chosen and applied if this flag is raised.
We give a few examples:

$l(x) := log(x), \quad D_{log} = (0, +\infty)$,
$log((0, 2]) = (-\infty, log(2)]$.
But also
$log([-5, 2]') = log((0, 2]) = (-\infty, log(2)]$.
The flag *domain overflow* should be set. It informs the user that the function has been evaluated for the intersection $\boldsymbol{x}' := \boldsymbol{x} \cap D_f = [-5, 2] \cap (0, +\infty) = (0, 2]$.

$h(x) := sqrt(x), \quad D_{sqrt} = [0, +\infty)$,
$sqrt([1, 4]) = [1, 2]$,
$sqrt([4, +\infty)) = [2, +\infty)$.
$sqrt([-5, -1]) = \varnothing$, an error message *sqrt* not defined for $[-5, -1]$, may be given to the user.
$sqrt([-5, 4]') = sqrt([0, 4]) = [0, 2]$.
The flag *domain overflow* should be set. It informs the user that the function has been evaluated for the intersection $\boldsymbol{x}' := \boldsymbol{x} \cap D_f = [-5, 4] \cap [0, +\infty) = [0, 4]$.

$k(x) := sqrt(x) - 1, \quad D_k = [0, +\infty)$,
$k([-4, 1]') = k([0, 1]) = sqrt([0, 1]) - 1 = [-1, 0]$.
The flag *domain overflow* should be set. It informs the user that the function has been evaluated for the intersection $\boldsymbol{x}' := \boldsymbol{x} \cap D_f = [-4, 1] \cap [0, +\infty) = [0, 1]$.

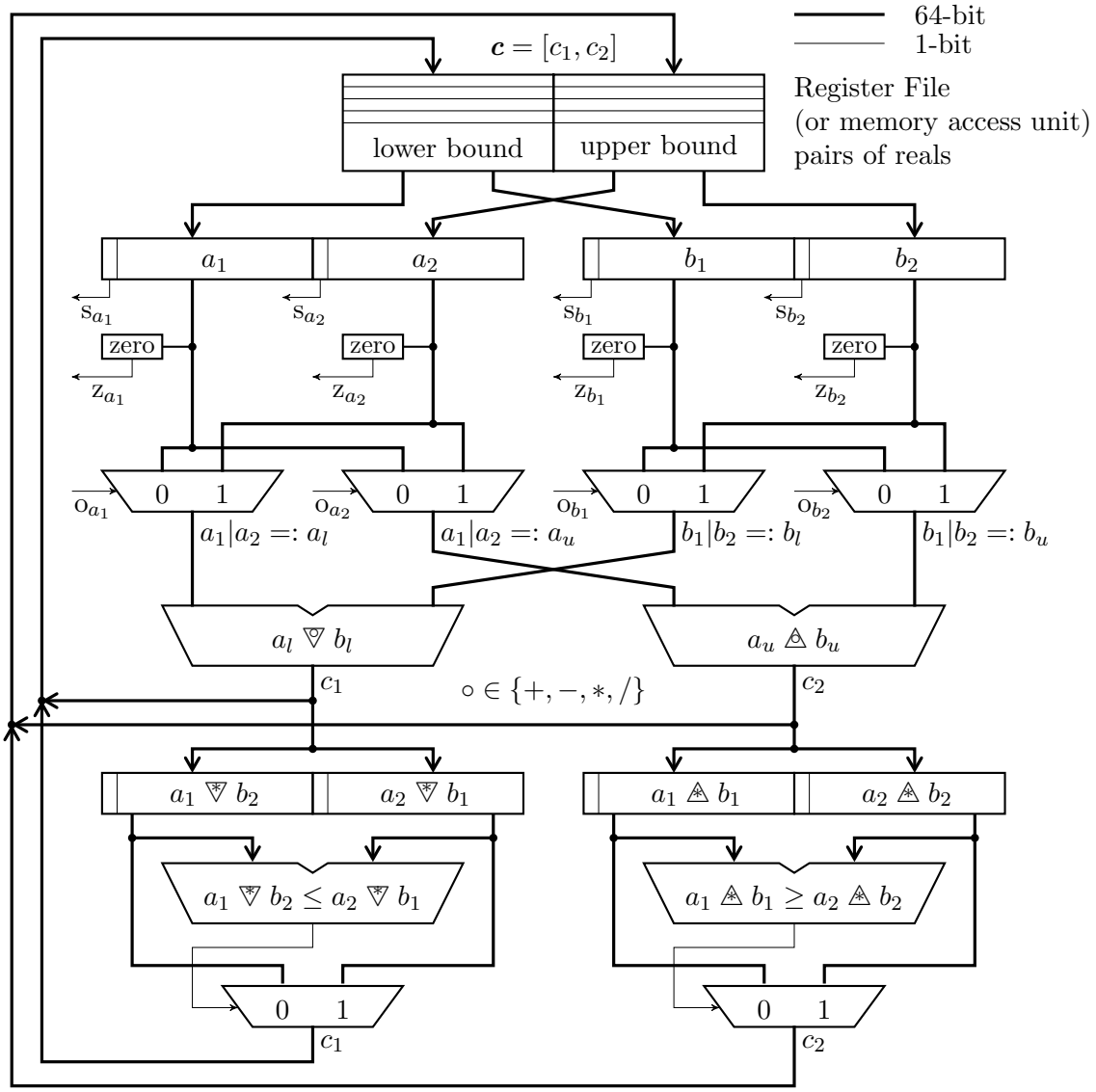
7. Hardware Support for Interval Arithmetic

In the early paper on interval arithmetic (1958) by Teruo Sunaga entitled: *Theory of an Interval Algebra and its Application to Numerical Analysis* the last sentence states: *A future problem will be: To revise the structure of the automatic digital computer from the standpoint of interval calculus and topology.* So the requirement to adapt the digital computer to the needs of interval arithmetic is as old as interval arithmetic itself. A solution to the problem is not given in Sunaga's paper. At the time of the paper the technology was poor. There was no hope of getting it realized on computers in those days.

The following figure gives a brief sketch of what hardware support for interval arithmetic may look like. It would not be hard to realize it in modern technology.

The circuitry broadly speaks for itself. The interval operands are loaded in parallel from a register file or a memory access unit. Then, after multiplexers have selected

the appropriate operands, the lower bound of the result is computed with rounding downwards and the upper bound with rounding upwards with the selected operands. In case of multiplication if both operands contain zero as an interior point a second multiplication is necessary. The result of both multiplications is forwarded to a comparison unit. Here for the lower bound of the result the lower and for the upper bound the higher of the two products is selected. This lower part of the circuitry could also be used to perfom comparison relations.



operands: $\boldsymbol{a} = [a_1, a_2]$, $\boldsymbol{b} = [b_1, b_2]$, result: $\boldsymbol{c} = [c_1, c_2]$.
s: sign, z: zero, o: operand select.

FIGURE 1. Circuitry for Interval Operations

Table 1 shows the control signals for the operand selection by the multiplexers. These signals are computed from the signs of the bounds of the interval operands $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$. In case of multiplication the signal ms is zero if only

one product pair is to be computed, and it is one if a second product pair is to be computed.

Every operand selector signal can be realized by two or three gates! For more details see [3] or [5].

OS	o_{a1}	o_{a2}	o_{b1}	o_{b2}
$+$	0	1	0	1
$-$	0	1	1	0
$*$	$s_{b2} + \overline{s_{a1}} \cdot s_{b1} + ms$	$\overline{s_{b1}} + \overline{s_{a1}} \cdot \overline{s_{b2}} + ms$	$\overline{ms}(s_{a2} + s_{a1} \cdot \overline{s_{b2}})$	$\overline{s_{a1}} + \overline{s_{a2}} \cdot \overline{s_{b1}} + ms$
$/$	$s_{b2} + s_{a1} \cdot s_{b1}$	$\overline{s_{b1}} + s_{a2} \cdot \overline{s_{b2}}$	$\overline{s_{a1}} + \overline{s_{a2}} \cdot s_{b1}$	$s_{a2} + s_{a1} \cdot s_{b1}$

TABLE 1. Operand selection signals.

The author of this paper is convinced that hardware support for interval arithmetic is absolutely necessary. The simpler a standard for interval arithmetic is kept the more likely it is that it will result in hardware support for interval arithmetic.

REFERENCES

[1] Alefeld, G., Herzberger, J.: *Introduction to Interval Computations*. Academic Press, New York, 1983.

[2] Kahan, W.: *A More Complete Interval Arithmetic*. Lecture Notes prepared for a summer course at the University of Michigan, June 17-21, 1968.

[3] Kirchner, R., Kulisch, U.: *Hardware support for interval arithmetic*. Reliable Computing 12:3, 225–237, 2006.

[4] Kulisch, U. W.: *Advanced Arithmetic for the Digital Computer – Design of Arithmetic Units*. Springer-Verlag, Wien, New York, 2002.

[5] Kulisch, U. W.: *Computer Arithmetic and Validity – Theory, Implementation and Applications*. De Gruyter, Berlin, New York, 2008.

[6] Kulisch, U. W.: *Complete Interval Arithmetic and its Implementation on the Computer*. To appear in the Proceedings of the Dagstuhl Seminar 08021, January 2008, Springer-Verlag, Wien, New York, 2009.

[7] Lohner, R.: *Interval Arithmetic in Staggered Correction Format*, 301–321, 1993. In *Scientific Computing with Automatic Result Verification*, Academic Press, San Diego, 1993, Adams, E., Kulisch, U. (eds.).

[8] IFIPWG-IEEE754R: *Letter of the IFIP WG 2.5 to the IEEE Computer Arithmetic Revision Group, 2007.*[7]

[9] Moore, R. E.: *Interval Analysis*. Prentice Hall Inc., Englewood Cliffs, New Jersey, 1966.

[10] Moore, R. E.: *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, Pennsylvania, 1979.

[11] Ratz, D.: *On Extended Interval Arithmetic and Inclusion Isotony.* Preprint, Institut für Angewandte Mathematik, Universität Karlsruhe, 1999.

[12] Rump, S. M.: *Kleine Fehlerschranken bei Matrixproblemen*. Dissertation, Universität Karlsruhe, 1980.

[7] See http://www.mathematik.uni-karlsruhe.de/ianm2/∼kulisch.