Motion divPair

J. Wolff von Gudenberg

April 26, 2013

Following motion 5 the basic arithmetic operations are defined as powerset operations.

$$\mathbf{a} \bullet \mathbf{b} = \{a \circ b | a \in \mathbf{a}, b \in \mathbf{b}, \text{if it is defined}\} \tag{1}$$

Only for the division $\mathbf{a}/\mathbf{b}$ when $\mathbf{b}$ contains zero in its interior the resulting sets are not necessarily intervals, there may be 2 semi-infinite closed or open intervals. We have 4 choices

(a) raise exception, terminate program

(b) split **b** in 2 and perform 2 divisions (motion 5)

(c) return a representation of the pair of unbounded intervals describing the exact solution set

(d) return the convex hull of the exact solution set; this is the P1788 division evaluation

Whereas (a) does not correspond to the paradigm of exception free computing (b) and (c) have the problem that the result is not an interval.

Hence (d) is the choice we have already voted for.

new operations

But especially for the interval Newton method, we want a division that computes enclosures for two half bounded, disjoint closed or open intervals, if appropriate. This is also useful for the reverse multiplication operation `mulRev` computing enclosures for the 2 parts of the solution set of the equation $b * x = a$.

$$C = \{x \in \mathbb{R} | \exists b \in \mathbf{b}, \exists a \in \mathbf{a}, b * x = a\} \tag{2}$$

At level 1 it suffices for the division, to proceed like motion 5 (see(b)). That means, if $\underline{b} < 0 < \overline{b}$, split the interval $\mathbf{b}$ at 0, and perform 2 divisions. The information returned by mulRevPair can be obtained by two ternary mulRev operations:

$mulRev(b, a, [-\infty, 0])$ and $mulRev(b, a, [0, +\infty])$

We want P1788 to provide a function that keeps the 2 quotients together.

We, therefore, move that P1788 shall have a version of the division and of the reverse multiplication operation, each of which returns 2 intervals. Only if zero is in the interior of **b**,two relevant closed intervals are produced. Otherwise the solution is enclosed by the first interval, the second output will be the empty set.

$$
\begin{aligned}
divPair : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} &\to \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\overline{a}/\overline{b}],[\overline{a}/\underline{b},\infty]) && \text{if} \quad (\overline{a}<0)\wedge(\underline{b}<0<\overline{b}) && (3)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\underline{a}/\underline{b}],[\underline{a}/\overline{b},\infty]) && \text{if} \quad (\underline{a}>0)\wedge(\underline{b}<0<\overline{b}) && (4)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\mathbf{a}/\mathbf{b},\emptyset) && \text{otherwise} && (5)
\end{aligned}
$$

$$
\begin{aligned}
mulRevPair : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} &\to \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \\
(\mathbf{b},\mathbf{a}) &\mapsto ([-\infty,\overline{a}/\overline{b}],[\overline{a}/\underline{b},\infty]) && \text{if} \quad (\overline{a}<0)\wedge(\underline{b}<0<\overline{b}) && (6)\\
(\mathbf{b},\mathbf{a}) &\mapsto ([-\infty,\underline{a}/\underline{b}],[\underline{a}/\overline{b},\infty]) && \text{if} \quad (\underline{a}>0)\wedge(\underline{b}<0<\overline{b}) && (7)\\
(\mathbf{b},\mathbf{a}) &\mapsto (mulRev(\mathbf{b},\mathbf{a}),\emptyset) && \text{otherwise} && (8)
\end{aligned}
$$

For infinite bounds use the general rules for infinities, but with $\infty/\infty = \infty$.

decorations

As a division divPair shall have a version for decorated intervals setting the local decoration to *trv*, if the denominator contains 0, and to *dac* or *com*, if not. mulRevPair has no decorated version, because it is a reverse operation.