Algebraic and Relational Concepts for Automated Interval Reasoning

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OUTLINE

- Problem statement
- Subsets and intervals in a lattice
- Interval reasoning and binary relation algebra
- Results and future works

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PROBLEM STATEMENT

- Get a further insight into modal intervals

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- Handling and Solving quantified interval Equations

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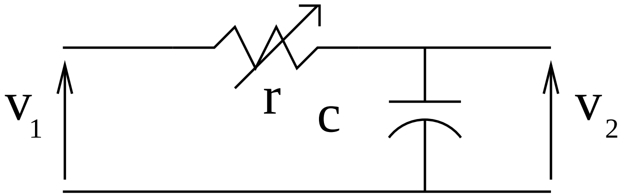
- Handling and Solving quantified interval Equations
  - Summary of the needs from an example

THE LOW PASS FILTER EXAMPLE

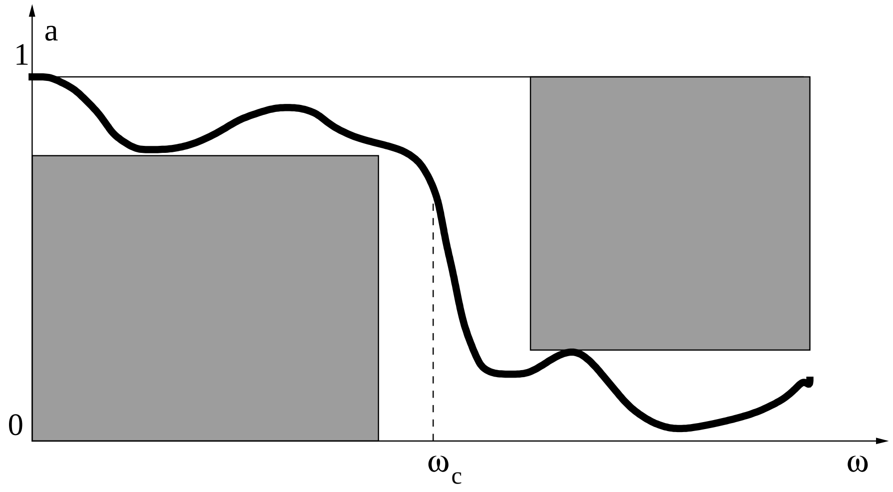


Two possible views:

- Black box : $a = \left|\frac{v_2}{v_1}\right| = f(\omega)$ where ω is the pulsation

- Design problems: $a = \left|\frac{v_2}{v_1}\right| = h(r, c, \omega)$

THE LOW PASS FILTER: BLACK BOX VIEW



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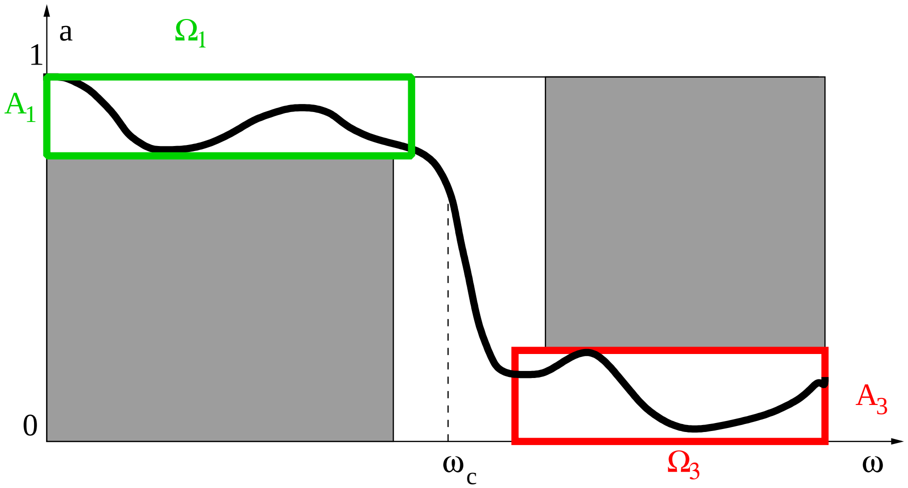

Passing/Rejection band $\forall \omega \in \Omega_1 \exists a \in A_1 \; a = f(\omega)$ $\forall \omega \in \Omega_3 \exists a \in A_3 \; a = f(\omega)$

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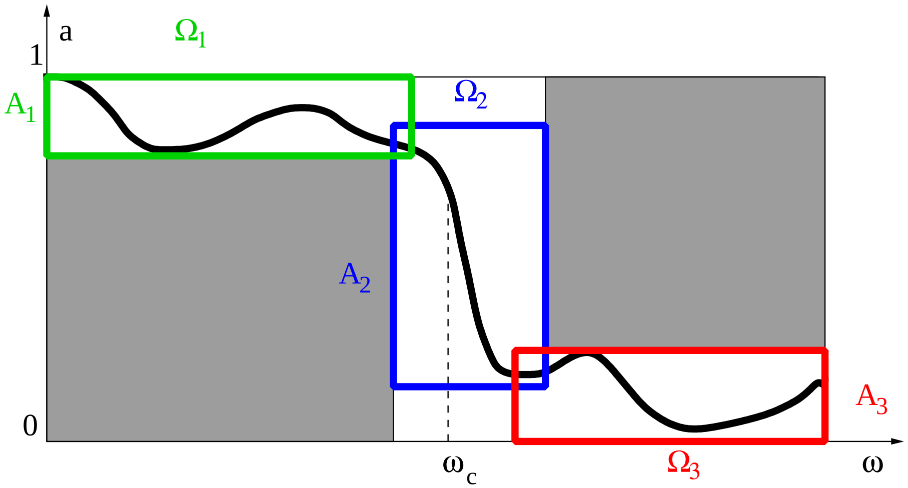

Passing/Rejection band $\forall \omega \in \Omega_1 \exists a \in A_1 \; a = f(\omega)$ $\forall \omega \in \Omega_3 \exists a \in A_3 \; a = f(\omega)$

Transition band $\forall a \in A_2 \exists \omega \in \Omega_2 \; a = f(\omega)$

THE LOW PASS FILTER: DESIGN PROBLEMS

1. Calculate the effects of the components r and c dispersion on the passing band. Given R, C and Ω_1, find A_1 such that:

$$\forall r \in R \forall c \in C \forall \omega \in \Omega_1 \exists a \in A_1 a = h(r, c, \omega)$$

2. Compensate the effects of the dispersion of the c values by a appropriate choice of r. Given C , A_1, and Ω_1 , find R such that:

$$\forall c \in C \exists r \in R \forall \omega \in \Omega_1 \exists a \in A_1 a = h(r, c, \omega)$$

SUMMARY OF THE NEEDS

1. Solve quantified interval equation
   - Partially addressed by interval arithmetic
   - Partially addressed by (modal) interval arithmetic
   - Fully addressed by binary relation algebra

2. Handle interval sets
   - Interval Lattice structure required

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ORDERED SETS, LATTICES AND INTERVALS

- (Partially) Ordered Set : Pair $(S, \leq)$.
  - Intervals: Nonempty subsets $[a,\ b]$ of S . (Interval set $I(S)$).
  - Problem: Many nonempty subsets are not described by (as) intervals
- Lattice = Ordered set + Algebraic structure
  - Any pair (a, b) has a greatest minorant inf(a, b) or $a \cap b$
  - Any pair (a, b) has a least majorant sup(a, b) or $a \cup b$

ORDERED SETS, LATTICES AND INTERVALS (CONTINUED)

- Finite Lattices
  - Any nonempty subset has a greatest minorant
  - Any nonempty subset has a least majorant

- Infinite lattices: the completion is required
  - Counter example 1: $S_1 = \{x \in \mathbb{Q} | x^2 \leq 2\}$
  - Counter example 2: $S_2 = \{x \in \mathbb{R} | x \leq 2\}$

LATTICES AND INTERVAL: DESCRIPTIONS OF SUBSETS

Given a complete lattice $(S, \leq)$, any nonempty subset S_1 can be described by the following intervals:

- The outer hull $H(S_1) = [\inf(S_1), \sup(S_1)]$.

- A "broadening" of any element $x_0 \in S_1$ defined as $W(S_1, x_0) = [\sup(S'_1(S_{1,}, x_0)), \inf(S''{}_1(S_{1,}, x_0))]$ where $S'{}_1(S_1, x_0) = \{x \in S | x \notin S_1 \wedge x \leq x_0\}$ and $S''{}_1(S_1, x_0) = \{x \in S | x \notin S_1 \wedge x \geq x_0\}$.

AN EXAMPLE: DESCRIPTIONS OF INTERVAL SETS

- Interval $[a, b]$ are subsets of the the incomplete lattice of real numbers $(\mathbb{R}, \leq)$
  - No completion to keep the field structure of $\mathbb{R}$

- Order relation $\leq_I$ between intervals defined

$$\leq_I = \{([a, b], [c, d]) \in I(\mathbb{R}) | a \leq c \wedge b \leq d\}$$

- The pair $(I(\mathbb{R}), \leq_I)$ define an incomplete lattice.

AN EXAMPLE: THE LATTICE OF INTERVALS(Continued)

- The completion of $(I(\mathbb{R}), \leq_I)$ is achieved by adding the folowing intervals

  1. $[-\infty, -\infty]$ as the infinum of $I(\mathbb{R})$.

  2. $[+\infty, +\infty]$ as the supremum of $I(\mathbb{R})$.

  3. $[-\infty, b]$ as the infinum of $S_2 = \{[x, b] \in I(\mathbb{R})\}$

  4. $[c, +\infty]$ as the supremum of $S_2 = \{[c, x] \in I(\mathbb{R})\}$

  5. $[-\infty, +\infty]$ as the supremum of $S_3 = \{[-\infty, b]\}$ and the infinum of $S_4 = \{[c, +\infty]\}$

INTERVAL OF THE SECOND KIND(*ISK*)

- The completed lattice can used for interval sets descriptions.

- This involves intervals of intervals (Intervals of the second kind)

$$[A, B] = \{X \in I(\mathbb{R}) | A \leq_I X \leq_I B\}$$

- Most of the ISK can be written as $[[a, b], [c, d]]$ where a, b, c and d are real numbers such that $a \leq b \leq d \wedge a \leq c \leq d$. The relative order of $b$and c is unspecified.

SOME REMARKABLE ISK

Example#1: ISK of the form $[[a,a],[d,d]]$. Given a predicate $p(x)$ such that $\forall x \; x \in [a,d] \supset p(x)$ we have $[[a,a],[d,d]] = \{X \in I(R) | [a,a] \leq_I X \leq_I [d,d] \wedge (\forall u \; u \in X \supset p(u))\}$.

Example#2: ISK of the form $[[a,b],[c,d]]$ with $b \leq c$. Given a predicate $p(x)$ such that $\forall x \; x \in [b,c] \supset p(x)$ we have $[[a,b],[c,d]] = \{X \in I(R) | [a,a] \leq_I X \leq_I [d,d] \wedge (\exists u \; u \in X \wedge p(u))\}$.

Example#3: ISK of the form $[[a,b],[c,d]]$ with $c \leq b$. Given a predicate $p(x)$ such that $\exists x \; x \in [c,b] \wedge p(x)$ we have $[[a,b],[c,d]] = \{X \in I(R) | [a,a] \leq_I X \leq_I [d,d] \wedge (\exists u \; u \in X \wedge p(u))\}$.

MODAL ISK AND MODAL INTERVALS

Definition#1: An ISK is said modal if it can written as $[[-\infty, b][c, +\infty]]$.

Definition#2: The inclusion relation between ISK is defined by

$$X \subseteq_{ISK} Y =_{def} \forall x \; x \in X \supset x \in Y$$

Definition#3: A modal interval is an interval $[a, b]$ where the $a \leq b$ condition is relaxed.

Definition#4: The modal interval inclusion relation is defined by

$$[a, b] \subseteq_M [c, d] =_{def} a \geq c \wedge b \leq d$$

MODAL ISK AND MODAL INTERVAL(End)

Property#1: The set of modal ISK a complete lattice

Property#2 The set of modal intervals is a complete lattice

Property#3: Both lattices are isomorphic.

Remark#1: ISK generalize modal intervals

Remark#2: In the present context no correspondance can be found between modal intervals and predicate set.

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INTERVAL REASONING AND BINARY RELATION ALGEBRA

- Three views for Binary Relation Algebra(BRA)
  - Purely abstract algebraic structure
  - Logical framework for reasoning
  - Representation in the real (mathematical!) world

BRA: ALGEBRAIC VIEW

- A BRA is a General algebra $\mathcal{A}(A, \cap, \cup, -, \odot, ^c, \mathbf{0}, \mathbf{1}, \mathbf{1}')$ over a set A.

- It includes a finite boolean lattice $\mathcal{A}(SA, \cap, \cup, -, \mathbf{0}, \mathbf{1})$. This lattice induces an order relation on S.

- Atoms a_i are the minimal elements of $S \backslash \{\mathbf{0}\}$. Any element r of S can be written $r = \bigcup_i (r \cap a_i)$.

- The algebra is completly defined given the values of the atomic expressions $(a_i \odot a_j) \cap a_k$.

BRA:LOGICAL VIEW

One defines a bijection between the algebraic expressions/equations and logical formulas/theorems

- To each algebraic a_i one associates some predicate $a_i(x, y)$ symbol with arity 2

- To each algebraic expression one associates a logical formula including *at most three variables*

- To each algebraic equation one associates a theorem *in a three variables logic*.

BRA: LOGICAL VIEW(Illustrations)

Example#1 We have $\in^c \odot p \Longleftrightarrow \exists z \ z \in x \wedge p(z,y)$

Example#2 We have $-(\in^c \odot(-p)) \Longleftrightarrow \forall z \ z \in x \supset p(z,y)$

Counterexample: The infinum lattice condition

$$\forall x \forall y \exists z((z \leq x \wedge z \leq y) \wedge (\forall u(u \leq x \wedge u \leq y) \supset u \leq z))$$

has no algebraic correspondant since it involves four variables.

BRA: LOGICAL VIEW(Question)

The logical formulas involved in the low pass filter are similar to

$$\forall c \in C \exists r \in R \forall \omega \in \Omega_1 \exists a \in A_1 a = h(r, c, \omega)$$

This formula involve eight variables. Can we write this formula as a three variables formula? The answer is yes. This can be shown using the third BRA (representation) view.

BRA: REPRESENTATION VIEW

Given a set S and an equivalence relation $\mathbf{1}_S$ over S

- The set $R(\mathbf{1}_S)$ of the binary relations over S included in $\mathbf{1}_S$ composed with the usual relation operations sastisfy the BRA axioms.

- A representation of an abstract algebra $\mathcal{A}(S, \cap, \cup, -, \odot, ^c, \mathbf{0}, \mathbf{1}, \mathbf{1}')$ over $\mathbf{1}_S$ is a subalgebra of $R(\mathbf{1}_S)$ isomorphic to $\mathcal{A}(S, \cap, \cup, -, \odot, ^c, \mathbf{0}, \mathbf{1}, \mathbf{1}')$.

BRA: REPRESENTATION VIEW : (Continued)

- The relation algebra generated over $I(\mathbb{R})$ by the relation $\in$ is a representation of some abstract BRA A

- This algebra can represented over the set of interval $n-$uples by n representations corresponding to each component. This give rise the n different relations $\in^{(1)} \dots \in^{(n)}$.

- This allows working with interval $n-$uples instead of interval. A carefull analysis show that only three variables are needed.

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INTERVAL REASONING

- We have elaborated a very general framework for interval reasoning

- Within this framework we can retrieve the results using modal interval analysis.

- This results can be generalized within the same framework

INTEREST AND WEAKNESS

- The main interest of our approach lies into the use of declarative reasoning rather arithmetic.

- The main veakness is the use of a function table. This difficulty shoud be overcome for rational functions by:

  1. Use of a analytic expression of the function
  2. Use of interval quantified table for some primitive functions (addition, product, ..)