Comments on Motion 5

G. William Walster

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1 First:

The first line under **Denotations** on page 1 is:

> $\mathbb{R}$: the set of real numbers, $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$,

The first sentence in the next paragraph is:

> Interval arithmetic over the real numbers $\mathbb{R}$ deals with closed and connected sets of real numbers.

Lines 4-6 contains the following sentences and footnote:

> If a bound is $-\infty$, or $+\infty$ the bound is not an element of the interval. Such intervals may also be written as $(-\infty, a]$, $[b, +\infty)$ or $(-\infty, +\infty)$ with a, $b \in \mathbb{R}$. They are also closed[2] intervals.
>
> [2]A subset of $\mathbb{R}$ is called closed if its complement is open.

Is $\mathbb{R}^* = [-\infty, +\infty]$ the 2-point compactification of $\mathbb{R}$? If so, does confining interval arithmetic to be "over the real numbers $\mathbb{R}$ with closed and connected sets of real numbers" imply that intervals with $-\infty$ or $+\infty$ as endpoints not compact?

How is it that the intervals $(-\infty, a]$, $[b, +\infty)$ and $(-\infty, +\infty)$ can be called "closed" because their complements are open? What is the universal set with respect to which complements are taken? If it is $\mathbb{R}^*$, then $(-\infty, a]^c = \{-\infty\} \cup (a, +\infty]$, $[b, +\infty)^c = [-\infty, b) \cup \{+\infty\}$, and $(-\infty, +\infty)^c = \{-\infty, +\infty\}$, none of which are open sets.

On the other hand, if the universal set with respect to which complements are taken is $\mathbb{R}$, then it appears as if new notation is required to reflect this fact. One option is to let "$(-\infty$" and "$+\infty)$" denote the lower and upper limits of real numbers. Then relative to the "closed" universal set $\mathbb{R} = [(-\infty, +\infty)]$, $\mathbb{R}^c = \emptyset$, which is both open and closed. However, $[(-\infty, a]^c = (a, +\infty)]$ and $[b, +\infty)]^c = [(-\infty, b)$, neither of which are "open".

The third and second paragraphs from the bottom of page 2 contain:

> Since in real analysis division by zero is not defined, the result of division by the interval $\mathbf{b} = [0, 0]$ can only be the empty set $\emptyset$.
>
> Whenever the zero in $\mathbf{b}$ coincides with a bound of the interval $\mathbf{b}$ the result of the division [is] the limit process $\mathbf{b}_1 \to 0$ or $\mathbf{b}_2 \to 0$ respectively.

The fist paragraph on page 3 contains:

> For division the set $b_1 < 0 < b_2$ devolves into the two distinct sets $[b_1, 0)$ [4] and $(0, b_2]$ and division by the set $b_1 < 0 < b_2$ actually means two divisions.
>
> [4] Since division by zero does not contribute to the solution set it does not matter whether a parenthesis or bracket is used here.

So apparently, to appear to be consistent with the removal of $\mathbb{R}^*\backslash\mathbb{R}$ from the universal set, when division by 0 occurs, it is also removed. The rational given is that division by 0 is undefined in real analysis, so it must also be in interval analysis. A counter argument is that real analysis deals only with singleton sets, or singletons. This is the fundamental reason why division by zero must be undefined in real analysis. Interval analysis is an *extension* of real analysis; not a subordinate system. Because intervals are based on sets, the same reason division by 0 is undefined in real analysis does not necessarily carry over to interval analysis. For example, with the universal set being $\mathbb{R}^*$, with neither inconsistencies nor containment failures:

$$\begin{aligned} 1 \div 0 &= \mathbb{R}^*\backslash\mathbb{R} \\ &= \{-\infty, +\infty\}. \end{aligned}$$

Footnote 5 on page 3 reads:

> [5] Fixed point division has always yielded two results.

What are they?

The tables for Addition, Subtraction, and Division on page 4 appears to be missing some entries. They are the results of:

$$\begin{gathered} -\infty + (+\infty) \\ +\infty + (-\infty) \\ -\infty - (-\infty) \\ +\infty - (+\infty) \\ -\infty \div (-\infty) \\ -\infty \div (+\infty) \\ +\infty \div (-\infty) \\ +\infty \div (+\infty) \end{gathered}$$

Is, for example

$$-\infty + (+\infty) = \mathbb{R}?$$

If so, what is the derivation?

What is the result of evaluating the expression $+\infty \div (+\infty)$? Is this the same as $(1 \div (+\infty)) \times (+\infty)$? If so, then $+\infty \div (+\infty) = 0$ because elsewhere, $1 \div (+\infty) = 0$ and $0 \times (+\infty) = 0$. A much more appealing answer is $+\infty \div (+\infty) = (0, +\infty)$.

So, there appear to be a number of gaps, inconsistencies, and/or open questions in the proposal. The root cause of these difficulties appears to lie in failing to use $\mathbb{R}^*$ as the universal set.

1.0.1 Second

If the motion is adopted as the *definition* of floating-point interval arithmetic operations, what will happen when a provably valid alternative is found that produces more narrow interval results? Will such a system be outlawed because it is inconsistent with the proposed motion in the then "interval standard", even if Motion 5 can be salvaged? If the answer is "No.", then after removing all its difficulties (which I do not believe is possible), I could support the motion. Otherwise, I must continue to vote "No." because its adoption will preclude future "quality of implementation features" from being sought, discovered, and/or implemented.