REVISION OF MOTION 13

COMPARISON RELATIONS FOR AN INTERVAL ARITHMETIC STANDARD

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Abstract. This proposal deals with the definition of comparison relations for an interval arithmetic computation standard and it considers their most important properties. Simplicity is a general requirement for this standard. So the number of comparison relations defined by the standard should be kept to a minimum. This proposal considers seven elementary comparison relations for intervals over the floating-point numbers.

Key words: computer arithmetic, floating-point arithmetic, interval arithmetic, arithmetic standards.

1. Introduction

Diverse programming environments for interval arithmetic have been developed during the last four decades. Several of these define a large number of comparison relations. This raises the question which of these relations should be required and specified by an interval arithmetic standard. Simplicity is a general requirement for this standard. So the number of comparison relations defined by the standard should be kept to a minimum. This is not a severe restriction. Comparison relations for intervals are easy to define and easy to execute. So if a user needs a particular relation for his application an ad hoc definition will suffice. The saying *a good theory is the best practice* may help making the right selection. From a theoretical point of view three comparison relations for intervals are fundamental for applications of interval arithmetic. They are: *equality, set inclusion*, and *less than or equal*. So this proposal takes particular care of these three relations. Their most important properties are considered. Explicit formulas for interval operations over the real and complex numbers and for intervals of vectors and matrices over the real and complex numbers can be developed by using these three comparison relations. See [4].

To a certain extent selecting the basic comparisons is a matter of taste. To keep the standard simple the original proposal required only 4 relations as elementary comparisons. Many others can be put together using these 4 relations.

Several very interesting comments were made by a number of colleagues. One by Nate Hayes [2] convinced me to revise the motion. In a mail of April 11 he suggests to extend the 4 comparisons in the original proposal by 3 additional relations. Indeed many derived comparisons can be expressed simpler by the 7 relations than by the original 4. In particular Nate Hayes shows in Table 2 of [2] that all certainly and possibly relations of Sun's interval Fortran can be expressed by very simple expressions with the 7 suggested relations.

So the revised Motion 13 now contains 7 comparisons as elementary relations for intervals. In this revised motion they are shown in the same sequence and expressed by the same symbols as in [2].

In the following $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$. With the order relation $\leq$, $\{\mathbb{R}^*, \leq\}$ is a complete lattice, i.e., every subset has an infimum and a supremum. $\mathbb{F}$ denotes the set of finite floating-point numbers of a given format and encoding and $\mathbb{F}^* := \mathbb{F} \cup \{-\infty, +\infty\}$. $\mathbb{IR}$ denotes the set of nonempty, closed and bounded real intervals and $\overline{\mathbb{IR}}$ the set of closed and connected real intervals. $\varnothing \in \overline{\mathbb{IR}}$. $\mathbb{IF}$ denotes the subset of $\mathbb{IR}$ with bounds of $\mathbb{F}$. The subset of all bounded or unbounded intervals of $\overline{\mathbb{IR}}$ with bounds of $\mathbb{F}$ is denoted by $\overline{\mathbb{IF}}$, $\varnothing \in \overline{\mathbb{IF}}$.

2. Comparison Relations and Lattice Operations

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are intervals of $\overline{\mathbb{IF}}$ with bounds $a_1 \le a_2$ and $b_1 \le b_2$ respectively, then these 7 relations are defined by:

$$
\begin{array}{lcll}
\boldsymbol{a} = \boldsymbol{b} & :\Leftrightarrow & a_1 = b_1 \wedge a_2 = b_2 & \boldsymbol{a} \text{ equals } \boldsymbol{b} \\
\boldsymbol{a} \subseteq \boldsymbol{b} & :\Leftrightarrow & b_1 \le a_1 \wedge a_2 \le b_2 & \boldsymbol{a} \text{ is a subset of } \boldsymbol{b} \\
\boldsymbol{a} \le \boldsymbol{b} & :\Leftrightarrow & a_1 \le b_1 \wedge a_2 \le b_2 & \boldsymbol{a} \text{ is less or equal to } \boldsymbol{b} \\
\boldsymbol{a} \preccurlyeq \boldsymbol{b} & :\Leftrightarrow & a_2 \le b_1 & \boldsymbol{a} \text{ precedes or touches } \boldsymbol{b} \\
\boldsymbol{a} \subset \boldsymbol{b} & :\Leftrightarrow & b_1 < a_1 \wedge a_2 < b_2 & \boldsymbol{a} \text{ is interior to } \boldsymbol{b} \\
\boldsymbol{a} < \boldsymbol{b} & :\Leftrightarrow & a_1 < b_1 \wedge a_2 < b_2 & \boldsymbol{a} \text{ is strictly less than } \boldsymbol{b} \\
\boldsymbol{a} \prec \boldsymbol{b} & :\Leftrightarrow & a_2 < b_1 & \boldsymbol{a} \text{ precedes } \boldsymbol{b}
\end{array}
$$

Since bounds for intervals of $\overline{\mathbb{IF}}$ may be $-\infty$ or $+\infty$ these comparison relations are executed as if performed in the lattice $\{\mathbb{F}^*, \le\}$.

Equality is an equivalence relation, *set inclusion* and *less than or equal* are order relations. These three relations are most fundamental for all interval computations.

With the relation $\subseteq$, $\{\overline{\mathbb{IF}}, \subseteq\}$ is a lattice. The least element in $\{\overline{\mathbb{IF}}, \subseteq\}$ is the empty set $\varnothing$ and the greatest element is the set $\mathbb{R} = (-\infty, +\infty)$. The infimum of two elements $\boldsymbol{a}, \boldsymbol{b} \in \overline{\mathbb{IF}}$ is the intersection and the supremum is the interval hull (convex hull):

$$
\begin{array}{rcl}
inf(\boldsymbol{a}, \boldsymbol{b}) = & \boldsymbol{a} \cap \boldsymbol{b} & := [max(a_1, b_1), min(a_2, b_2)] \quad \text{or the empty set } \varnothing, \\
sup(\boldsymbol{a}, \boldsymbol{b}) = & \boldsymbol{a} \overline{\cup} \boldsymbol{b} & := [min(a_1, b_1), max(a_2, b_2)].
\end{array}
$$

The intersection of an interval with the empty set is the empty set. The interval hull with the empty set is the other operand.

With the order relation $\le$, $\{\overline{\mathbb{IF}}, \le\}$ is also a lattice. The *greatest lower bound* (glb) and the *least upper bound* (lub) of $\boldsymbol{a}, \boldsymbol{b} \in \overline{\mathbb{IF}}$ are the intervals

$$
\begin{array}{rl}
glb(\boldsymbol{a}, \boldsymbol{b}) & := [min(a_1, b_1), min(a_2, b_2)], \\
lub(\boldsymbol{a}, \boldsymbol{b}) & := [max(a_1, b_1), max(a_2, b_2)],
\end{array}
$$

respectively. The least element of $\{\overline{\mathbb{IF}}, \le\}$ is the interval $(-\infty, minreal]$ and the greatest element is $[maxreal, +\infty)$. The greatest lower bound and the least upper bound of an interval with the empty set are both the empty set.

If in the formulas for $glb(\boldsymbol{a}, \boldsymbol{b})$, $lub(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{a} \cap \boldsymbol{b}$, $\boldsymbol{a} \overline{\cup} \boldsymbol{b}$, a bound is $-\infty$ or $+\infty$ a parenthesis may be used for this interval bound to denote the resulting interval. This bound is not an element of the interval.

If in any of the first three comparison relations in the above table both operands are the empty set, the result is true. If in $\boldsymbol{a} \subseteq \boldsymbol{b}$ or $\boldsymbol{a} \subset \boldsymbol{b}$, $\boldsymbol{a}$ is the empty set the result is true. Otherwise the result is false if in any of the 7 comparison relations any operand is the empty set.

A particular case of inclusion is the relation *element of*. For $a \in \mathbb{F}$ it is defined by

$$a \in \boldsymbol{b} :\Leftrightarrow b_1 \le a \wedge a \le b_2.$$

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