

Motion 12: Inner addition/subtraction over intervals

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1 Introduction

This paper specifies the operations for inner addition and subtraction over intervals for the forthcoming IEEE interval standard [1].

These operations can be used e. g. for convenient presentation of the solutions of certain interval algebraic equations. For example the inner difference of the intervals A, B is the solution of the equation $B + X = A$ when this solution exists (i. e. when $\text{width}(A) \leq \text{width}(B)$), or is the solution of the equation $A - X = B$ when the solution exists (i. e. when $\text{width}(A) \geq \text{width}(B)$). Many applications are related to ranges of monotone functions or the control of accuracy of interval-arithmetic computational results. Some of these applications are briefly outlined in Section 3.

2 Inner addition and subtraction over intervals

In real interval-arithmetic inner addition and inner subtraction over two intervals $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathbb{IR}$ are defined as:

$$A +^- B = \begin{cases} [a^- + b^+, a^+ + b^-], & \text{if } w(A) \geq w(B), \\ [a^+ + b^-, a^- + b^+], & \text{otherwise.} \end{cases} \quad (1)$$

$$A -^- B = \begin{cases} [a^- - b^-, a^+ - b^+], & \text{if } w(A) \geq w(B), \\ [a^+ - b^+, a^- - b^-], & \text{otherwise.} \end{cases} \quad (2)$$

where $w(A) = a^+ - a^-$ is the width of A .

Remark. Note that (1), (2) are always defined and thus they are “operations” in algebraic sense (and not partial operations).

Inner addition and inner subtraction are related by $A +^- B = A -^- (-B)$, $A -^- B = A +^- (-B)$, where $-B = (-1) * B$.

In accordance to Motion 5 [3] outward digital roundings shall be available for the above operations, that is: $\Diamond(A +^- B)$, $\Diamond(A -^- B)$.

Remark. The outward rounded inner operations $\Diamond(A +^- B)$, $\Diamond(A -^- B)$ can be defined in the spirit of [37] as follows:

Inner operations --- outward rounding. There is an operation `innerAdditionOut(xx,yy)` that returns for any two intervals $xx=[l,u]$ and $yy=[l',u']$ the tightest interval containing (the points) $l+u'$ and $u+l'$.

There is an operation `innerSubtractionOut(xx,yy)` that returns for any two intervals $xx=[l,u]$ and $yy=[l',u']$ the tightest interval containing (the points) $l-l'$ and $u-u'$.

Remark. Another type of digital roundings (named inward roundings) will be the subject of a future motion.

3 Rationale

In this section we consider two aspects on inner operations: the algebraic one and one for the presentation of functional ranges and computation with such ranges.

The operations for inner addition and subtraction over intervals are mentioned in [37], see p. 37, Section 5.6.(3). Both operations have been used in numerous applications, see e. g. [4]–[35].

3.1 Algebraic properties of inner operations

Usually Hukuhara difference [2] is defined in the set of convex bodies K as follows: Given any two sets $A, B \in K$, if there exists a set $X \in K$ satisfying $A = B + X$, then $X = A \ominus B$ is called the *Hukuhara difference* of the sets A and B . The Hukuhara difference plays an important role in the theory of convex bodies [39].

The Hukuhara difference can be symbolically expressed as follows:

$$A \ominus B = X \iff A + X = B. \quad (3)$$

Formula (3) shows that Hukuhara difference is the solution X of $A + X = B$ (whenever existing).

In the special case of one-dimensional intervals $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathbb{IR}$ the Hukuhara difference can be written as:

$$A \ominus B = \begin{cases} [a^- - b^-, a^+ - b^+], & \text{if } w(A) \geq w(B), \\ \text{not defined,} & \text{otherwise,} \end{cases} \quad (4)$$

wherein $w(A) = a^+ - a^-$ is the width of A .

Remark. Note that $A \ominus B$ is not an operation in the algebraic sense, but only a partial operation.

The *inner* operations for addition/subtraction of (one-dimensional) intervals from the present motion (1), (2) can be introduced using a similar “algebraic approach”. Thus the *inner difference* $A -^- B$, $A, B \in I(\mathbb{R})$, is the solution Z of a “linear” equation either of the type $B + Z = A$, or of the type $A - Z = B$ (depending on which one is solvable; one of the equations is always solvable).

For more clarity we shall next denote multiplication by -1 as $\neg A = (-1) * A = [-a^+, -a^-]$, resp. subtraction of intervals A, B as $A \neg B = A + (\neg B) = [a^- - b^+, a^+ - b^-]$.

Remark. Introducing the operation of inner subtraction makes notation $A - B$ vague — is this outer (standard) subtraction or inner? Therefore we shall avoid in the sequel the dubious notation $A - B$ using $A \neg B$ for the outer (standard) subtraction and $A -^- B$ for the inner one.

Symbolically we have

$$A -^- B = X \iff \begin{cases} B + X = A, & \text{if solution exists;} \\ A \neg X = B, & \text{if solution exists.} \end{cases} \quad (5)$$

An equivalent way to express the above is:

$$A -^- B = \begin{cases} Y|_{B+Y=A}, & \text{if } w(B) \leq w(A); \\ X|_{A \neg X=B}, & \text{if } w(A) \leq w(B). \end{cases} \quad (6)$$

Similarly inner addition can be introduced:

$$A +^- B = X \iff \begin{cases} \neg B + X = A, & \text{if solution exists;} \\ \neg A + X = B, & \text{if solution exists,} \end{cases} \quad (7)$$

which can be alternatively written as:

$$A +^- B = \begin{cases} Y|_{\neg B+Y=A}, & \text{if } w(B) \leq w(A), \\ X|_{\neg A+X=B}, & \text{if } w(A) \leq w(B). \end{cases} \quad (8)$$

Note that $w(\neg B) = w(B)$ so that the inequalities $w(B) \leq w(A)$ and $w(\neg B) \leq w(A)$ are equivalent. Inner addition and inner subtraction are related by $A +^- B = A -^- (\neg B)$, $A -^- B = A +^- (\neg B)$ as this easily follows from (6), (8).

Rules for algebraic transformations. For $A, B \in \mathbb{IR}$ denote $A +^+ B = A + B$, $A -^+ B = A \neg B = A + (\neg B)$. Then using the binary symbol $\sigma \in \{+, -\}$ we can write $A +^\sigma B$, $A -^\sigma B$.

Besides the sign functional σ we use $\phi : \mathbb{IR} \otimes \mathbb{IR} \rightarrow \{+, -\}$, defined as

$$\phi(A, B) = \begin{cases} +, & \text{if } w(A) \geq w(B); \\ -, & \text{otherwise,} \end{cases}$$

For $A = [a^-, a^+] \in \mathbb{IR}$ we define $A +^- B$ by:

$$A +^- B = [a^{-\gamma} + b^\gamma, a^\gamma + b^{-\gamma}], \quad \gamma = \phi(A, B), \quad (9)$$

We recall the “join” operation (denoted symbolically “ \vee ”) in the special case of real numbers. For $\alpha, \beta \in \mathbb{R}$, the join $\alpha \vee \beta \in \mathbb{R}$ is either the interval $[\alpha, \beta] \in \mathbb{IR}$ or the interval $[\beta, \alpha] \in \mathbb{IR}$ depending on whether $\alpha \leq \beta$ or $\alpha \geq \beta$.

Using join we have

$$A +^- B = [(a^- + b^+) \vee (a^+ + b^-)].$$

Note that $A +^- (\neg A) = 0$, meaning that $\neg A = [-a^+, -a^-]$ is inverse element w. r. t. the operation “ $+^-$ ”.

The operation $A -^- B = A +^- (\neg B)$ can be written

$$A -^- B = [a^{-\gamma} - b^{-\gamma}, a^\gamma - b^\gamma], \quad \gamma = \phi(A, B). \quad (10)$$

Using the operation “join”, the above can be written

$$A -^- B = [(a^- - b^-) \vee (a^+ - b^+)].$$

Let us consider some algebraic properties of inner operations addition/subtraction.

The two operations for addition $+$, $+^-$ can be considered as one operation in two modes (directions) to be denoted “ $+^\theta$ ”, wherein $\theta \in \{+, -\}$, and referred to as “directed addition”. For $\theta = +$ the operation $+^\theta$ is the standard (positively directed) addition, “ $+$ ”, whereas for $\theta = -$, $+^\theta$ is the nonstandard (negatively directed) addition, “ $+^-$ ”. The directed addition $+^\theta$ can be expressed:

$$A +^\theta B = [(a^- + b^{-\theta}) \vee (a^+ + b^\theta)].$$

Below we present some properties of directed/inner addition.

Commutativity of inner addition. For $A, B \in \mathbb{IR}$ we have $A +^- B = B +^- A$.

Conditional associativity of directed addition. Directed addition are conditionally associative in the sense that for each triple $A, B, C \in I(\mathbb{R})$ and each pair $\theta_1, \theta_2 \in \{+, -\}$, there exist another pair $\theta_3, \theta_4 \in \{+, -\}$, such that

$$(A +^{\theta_1} B) +^{\theta_2} C = A +^{\theta_3} (B +^{\theta_4} C).$$

Moreover, θ_3, θ_4 are simple functions of the widths of the intervals and can be easily computed.

$X = [0, 0] = 0$ is the unique neutral element with respect to inner addition $+^-$, that is for every $A \in \mathbb{IR}$

$$A = X +^- A = A +^- X \iff X = [0, 0].$$

Every element $A \in \mathbb{IR}$ has unique inverse w. r. t. “ $+^-$ ”, and this is the element $\neg A$.

Outer addition is commutative and associative but has no inverse, whereas inner addition is not associative but commutative and has inverse. Consedered together, as one “directed” operation in two different modes, we can say that

this directed operation is conditionally acossiative. So both modes complement each other.

For $p \in \mathbb{R}$ define $\sigma(p) = \{+, \text{ if } p \geq 0; -, \text{ if } p < 0\}$.

Quasidistributive law: For $A \in \mathbb{IR}, p, q \in \mathbb{R}$ and $*$ multiplication by scalars

$$(p + q) * A = p * A +^{\sigma(p)\sigma(q)} q * A, \quad (11)$$

Remark. Note the equality relation in (11); recall that the corresponding law formulated in classic operations only gives inclusion: $(\alpha + \beta) * C \subseteq \alpha * C + \beta * C$, whereas now we have $(\alpha + \beta) * C = \alpha * C +^{\sigma(\alpha)\sigma(\beta)} \beta * C$.

3.2 Inner operations and monotone functions

Let $X \in \mathbb{IR}$ and f, g be two continuous functions defined on $x \in X$.

For the functional ranges $f(X) = \{f(x) \mid x \in X\}$, $g(X) = \{g(x) \mid x \in X\}$ we have $(f + g)(X) \subseteq f(X) + g(X)$.

Moreover, we have $(f + g)(X) = f(X) + g(X)$, if f, g equally monotone and this is true for arbitrary equally monotone functions f, g .

This observation can be used to define the operation addition of two intervals $A, B \in \mathbb{IR}$ as follows:

Definition. Given two intervals $A, B \in \mathbb{IR}$ take any two equally monotone functions f, g defined on X , s. t. $f(X) = A$, $g(X) = B$. We then define the sum of the intervals $A, B \in \mathbb{IR}$ as $A + B = (f + g)(X)$.

The above definition is correct, since $(f + g)(X)$ depends only on the choice of A, B .

Example 1. If $X = [0, 1]$, $A = [a^-, a^+]$, $B = [b^-, b^+]$, we can choose the functions f, g to be the monotone increasing linear functions defined on $[0, 1]$ by:

$$f^+(\xi) = (1 - \xi)a^- + \xi a^+, \quad g^+(\xi) = (1 - \xi)b^- + \xi b^+,$$

or the monotone decreasing linear functions defined on $[0, 1]$ as:

$$f^-(\xi) = (1 - \xi)a^+ + \xi a^-, \quad g^-(\xi) = (1 - \xi)b^+ + \xi b^-.$$

Remark. In practice, for smooth functions f, g and for small interval argument X in “half” of the situations f, g are *equally* monotone functions and we have $A + B = (f + g)(X)$ (showing the usefulness of the operation interval addition). Now the question is what can be done in the other “half” of the cases when f, g are two *differently* monotone functions. In this situation denoting $f(X) = A$, $g(X) = B$, we only have $(f + g)(X) \subseteq f(X) + g(X) = A + B$, but the inclusion can be “rough”. Typical example is $f(x) = x$, $g(x) = -x$, then we have $0 = \{x + (-x) \mid x \in X\} \subseteq X + (-X) = X - X$, with $\omega(X - X) = 2\omega(X)$, showing how rough an inclusion can be.

Following arguments similar to the ones for the definition of addition we can proceed as follows.

Definition. Given $A, B \in \mathbb{IR}$, take any two differently monotone functions f, g s. t. $f(X) = A$, $g(X) = B$ and $f + g$ is monotone. Define “inner addition” by means of $A +^- B = (f + g)(X)$.

Obviously, $(f + g)(X)$ depends only on the choice of A, B so the above definition is correct.

The above can be summarized as follows.

Denote $CM(T)$ the set of continuous monotone functions on $T \in \mathbb{IR}$.

For $f \in CM(T)$ denote $\tau_f = \tau(f; T) \in \{+, -\}$, where

$$\tau(f; T) = \begin{cases} +, & \text{if } f \text{ is isotone in } T; \\ -, & \text{if } f \text{ is antitone in } T. \end{cases}$$

For $f, g \in CM(T)$, the equality $\tau_f = \tau_g$ means, that both f, g are isotone or both are antitone in T ; $\tau_f = -\tau_g$ means that one function is isotone and the other is antitone.

Proposition. ([29],[30]) Let $f, g \in CM(T)$. Then for every $X \subseteq T$ we have:

- i) $f + g \in CM(T)$ implies $(f + g)(X) = f(X) +^{\tau_f \tau_g} g(X)$,
- ii) $f - g \in CM(T)$ implies $(f - g)(X) = f(X) -^{\tau_f \tau_g} g(X)$.

3.3 Mid-rad presentation

The mid-rad presentation of intervals adds important further deep insight to the application of inner interval-arithmetic operations.

Denoting the midpoint and the radius of $A = [a^-, a^+] \in \mathbb{IR}$ resp. by a' and a'' , we have

$$a' = (a^- + a^+)/2, \quad a'' = (a^+ - a^-)/2.$$

The form $A = (a'; a'')$ is called mid-rad presentation. Conversely

$$a^- = a' - a'', \quad a^+ = a' + a''.$$

For addition and subtraction we have

$$\begin{aligned} A + B &= (a' + b'; a'' + b''), \\ A \neg B &= (a' - b'; a'' + b''). \end{aligned}$$

For inner addition and subtraction we have

$$\begin{aligned} A +^- B &= (a' + b'; |a'' - b''|), \\ A -^- B &= (a' - b'; |a'' - b''|). \end{aligned}$$

Remark. The above formulae illuminate the interval operations when radii are small, which corresponds to the “approximate number” aspect of intervals.

Using multiplication by -1 addition (inner addition) is representable by means of subtraction (inner subtraction), that is

$$A - B = A \neg B = A + (\neg B), \quad A +^- B = A -^- (\neg B).$$

Multiplication by -1 in mid-rad presentation is

$$\neg A = (-1) * A = (-1) * (a'; a'') = (a'; |a''|), \quad A \in \mathbb{IR}.$$

More generally we have for $\gamma \in \mathbb{R}$

$$\gamma * A = \gamma * (a'; a'') = (\gamma a'; |\gamma| a''). \quad (12)$$

Recall that multiplication by scalars in end-point presentation has the form:

$$\gamma * A = \begin{cases} [\gamma a^-, \gamma a^+], & \text{if } \gamma \geq 0, \\ [\gamma a^+, \gamma a^-], & \text{if } \gamma < 0. \end{cases} \quad (13)$$

Remark. Note that the coordinates in (12) are “separated”, which is not the case of multiplication by scalars in end-point presentation (13). Thus mid-rad presentation allows to reduce certain classes of linear interval problems to linear numerical problems.

References

- [1] IEEE interval standard <http://grouper.ieee.org/groups/1788>
- [2] Hukuhara, M., Integration des applications mesurables dont la valeur est un compact convexe, *Funkcialaj Ekvacioj*, 10 (1967), 205-223.
- [3] Arithmetic operations for floating-point intervals. Author: Ulrich Kulisch, Proposer: Bo Einarsson, Seconder: John Pryce. Motion 5, <http://grouper.ieee.org/groups/1788>
- [4] Angelov, R., S. Markov, Extended Segment Analysis, *Freiburger Intervall-Berichte* 81(10), Inst. f. Angewandte Mathematik, U. Freiburg i. Br., 1981, 1–63.
- [5] Bartholomew-Biggs, M., Zakovic, S.: Using Markov’s interval arithmetic to evaluate Bessel Functions, *Num. Algorithms*, 10, 1995, 261–287.
- [6] Dimitrova, N., Über die Distributivgesetze der erweiterten Intervallarithmetic. *Computing* 24, 33–49 (1980).
- [7] Dimitrova, N., S. Markov, On the interval-arithmetic presentation of the range of a class of monotone functions of many variables. In: *Computer arithmetic, scientific computation and mathematical modelling* (Ed. by E. Kaucher, S. Markov, G. Mayer), J. C. Balzer, Basel, 1991, 213-228.
- [8] N. Dimitrova, S. Markov: Über die intervallararithmetische Berechnung des Wertebereichs einer Funktion mit Anwendungen. *Freiburger Intervall-Berichte* 81/4, Univ. Freiburg, Germany, 1981, 1–22.
- [9] Dimitrova, N., S. M. Markov: Interval Methods of Newton Type for Nonlinear Equations. *Pliska stud. math. bulg.* 5, 1983, 105–117.
- [10] Dimitrova, N., S. Markov: Interval-arithmetic Algorithms for Simultaneous Computation of all Polynomial Zeroes. *Contributions to Computer Arithmetic and Self-Validating Numerical Methods*, C. P. Ullrich (Ed.), IMACS Annals on Computing and Applied Mathematics, J. C. Baltzer Publ., Basel, 7, 1990, 291–300.
- [11] Dimitrova, N., S. Markov: On the Interval-arithmetic Presentation of the Range of a Class of Monotone Functions of Many Variables. *Computer Aritmetic, Scientific Computation and Mathematical Modelling*,

- E. Kaucher, S. M. Markov, G. Mayer (Eds.), J. C. Baltzer Publ. IMACS, 12, 1991, 213–228.
- [12] Dimitrova, N., Markov, S., Popova, E., Extended Interval Arithmetics: New Results and Applications. In: Computer Arithmetic and Enclosure Methods (Eds. L. Atanassova, J. Herzberger), North-Holland, Amsterdam, 1992, 225–234.
 - [13] Dimitrova, N. S.: On Some Properties of an Interval Newton Type Method and its Modification. Computing, Suppl. 9, 1993, 21–33.
 - [14] Dimitrova, N. S., S. M. Markov: On Validated Newton Type Method for Nonlinear Equations. Interval Computations, 2, 1994, 27–51.
 - [15] Dimitrova, N. S.: On a Numerical Approach for Solving a Class of Nonlinear Systems. Scientific Computing and Validated Numerics, G. Alefeld, A. Frommer, B. Lang (Eds.), Akademie Verlag, Berlin, 1996, 147–153.
 - [16] Dimitrova, N. S.: On a Parallel Method for Enclosing Real Roots of Nonlinear Equations. In: I. T. Dimov, Bl. Sendov, P. S. Vassilevski (eds.), Numerical Methods and Applications, World Scientific, 1994, 78–84.
 - [17] Dimitrova, N. S., S. M. Markov: On Validated Newton Type Method for Nonlinear Equations. Interval Computations, 2, 1994, 27–51.
 - [18] Markov, S., Extended interval arithmetic. Compt. rend. Acad. bulg. Sci. 30, 9, 1977, 1239–1242.
 - [19] Markov, S., A non-standard subtraction of intervals. Serdica 3, 1977, 359–370.
 - [20] Markov, S., A differential calculus for interval-valued functions based on extended interval arithmetic. Compt. rend. Acad. bulg. Sci. 30, 10, 1977, 1377–1380.
 - [21] Markov, S., On the extended interval arithmetic. Compt. rend. Acad. bulg. Sci., 31, 2, 1978, 163–166.

- [22] Markov, S., Calculus for interval functions of real variable using extended interval arithmetic. *Compt. rend. Acad. bulg. Sci.* 31, 4, 1978, 373–376.
- [23] Markov, S., Existence and uniqueness of solutions of the interval differential equation $X' = f(t, X)$. *Compt. rend. Acad. bulg. Sci.* 31, 12, 1978, 1519–1522.
- [24] Markov, S., Calculus for interval functions of a real variable. *Computing* 22, 325–337 (1979).
- [25] Markov, S., N. Dimitrova, Rechengesetze der erweiterten Intervallarithmetic, *Freiburger Intervall-Berichte*, Univ. Freiburg, 79/10, (1979), 1–19.
- [26] Markov, S., Some applications of the extended interval arithmetic to interval iterations, *Computing Suppl.* 2, 1980, 69–84.
- [27] Markov, S., Interval Differential Equations, *Interval Mathematics 1980* (Ed. K. Nickel), Academic Press, 1980, 145–164.
- [28] Markov S., On the Interval Computation of Elementary Functions, *Comp. Rend. Acad. Bulg. Sci.* 34, 32, 319–322, (1981).
- [29] Markov, S., Extended Interval Arithmetic Involving Infinite Intervals. *Mathematica Balkanica, New Series* 6 (3), 1992, 269–304.
- [30] Markov, S., On the Presentation of Ranges of Monotone Functions Using Interval Arithmetic, *Interval Computations* 4 (6), 1992, 19–31.
- [31] Markov, S., On Directed Interval Arithmetic and its Applications, *J. UCS* 1 (7), 1995, 514–526.
- [32] Markov, S., On the foundations of interval arithmetic. In: Alefeld, G., A. Frommer, B. Lang (Eds.) *Scientific Computing and Validated Numerics*, *Mathematical Research* 90, Akademie Verlag, 307–313, 1996.
- [33] Markov, S., Isomorphic Embeddings of Abstract Interval Systems. *Reliable Computing* 3 (3), 1997, 199–207.

- [34] Nesterov, V., How to use Monotonicity-Type Information To Get Better Estimates of the Range of Real-Valued Functions, *Interval Computations* 4, 1993, 3–12.
- [35] Nesterov, V., Interval and Twin Arithmetics, *Reliable Computing* 3, 4, 369–380, 1997.
- [36] Neumaier, A., Computer graphics, linear interpolation, and nonstandard intervals, December 22, 2008 WWW: <http://www.mat.univie.ac.at/~neum>
- [37] Neumaier, A., Vienna proposal for interval standardization, Final version, December 19, 2008. <http://www.mat.univie.ac.at/~neum/papers.html#1788>
- [38] Neumaier, A., Improving interval enclosures, Manuscript (2008). <http://www.mat.univie.ac.at/~neum/papers.html#encl>
- [39] Schneider, R., *Convex Bodies: The Brunn-Minkowski Theory* (Cambridge Univ. Press, 1993).