Fuzzy Partial Order Relations for Intervals – for the reference of IEEE-1788

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1 Incomparability of Intervals in Binary Logic

For any two given real numbers x and y, based on their positions on the real line, the statement "x is less than y" can only be either true or false. However, the relation between two nonempty intervals $\mathbf{x}$ and $\mathbf{y}$ can be fairly complicated. They can be disconnected, partially overlapping, or completely overlapping. In [1], Allen listed 13 possible temporal relationships between 2 time intervals. Krokhin et al. further studied the relations in [4], and indicated that the relations between intervals could be $2^{13} = 8192$ possible unions of the 13 basic interval relations. This means that the statement "an interval $\mathbf{x}$ is less than another interval $\mathbf{y}$" cannot be expressed in binary logic. In short, there is not a general binary ordering relationship between two intervals. In [3], we defined a binary interval operator, $\prec$, to indicate the degree (or fuzzy membership) of an interval $\mathbf{x}$ less than another interval $\mathbf{y}$. Then we proved that the operator $\prec$ in fact establishes a fuzzy partial order relation for intervals. Here, we present it to the IEEE-1788 Interval Arithmetic Standard Committee. We use Kearfott's notation here. That is, a lowercase boldfaced letter represents an interval, and its lower and upper bounds are specified by an underline and an overline, respectively.

2 The $\prec$ operator between two intervals

Let $\mathbf{x}$ and $\mathbf{y}$ be two intervals. If $\forall x \in \mathbf{x}$ and $y \in \mathbf{y}$ we have $x < y$, then $\mathbf{x}$ is less than $\mathbf{y}$. This happens only when $\mathbf{x} \cap \mathbf{y} = \emptyset$ and $\mathbf{x}$ is completely on the left side of $\mathbf{y}$. In this case, we denote $\mathbf{x} \prec \mathbf{y} = 1$.

An interval $\mathbf{x}$ can be on the left of another interval $\mathbf{y}$ but partially overlapped (i.e., $\underline{x} \leq \underline{y} \leq \overline{x} < \overline{y}$). In this case, we may say that "$\mathbf{x}$ is weakly less than $\mathbf{y}$" and still denote $\mathbf{x} \prec \mathbf{y} = 1$ (or 1^- if one wants to specify the weakness).

Assume $\mathbf{x} \subset \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; thus, $\underline{x} \leq \underline{y} \leq \overline{x} \leq \overline{y}$ but $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$ cannot both be true simultaneously. It is easy to prove $0 \leq \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} \leq 1$ when $\mathbf{x} \subset \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Also, when $\underline{x} = \underline{y}$, $\dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} = 1$ and $\dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} = 0$ as $\overline{x} = \overline{y}$. Hence, we define

$$\mathbf{x} \prec \mathbf{y} = \frac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})}.$$

When the midpoints of $\mathbf{x}$ and $\mathbf{y}$ overlap (i. e., $m(\mathbf{x}) = m(\mathbf{y})$, and $w(\mathbf{x}) \neq w(\mathbf{y})$), we have $\dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} = 0.5$.

Finally, when $\mathbf{x}$ and $\mathbf{y}$ are the same, one is equally greater and less than the other, and we write $\mathbf{x} \prec \mathbf{y} = 0.5$.

Summarizing the above discussion, we define a binary operation with the operator $\prec$ for two intervals $\mathbf{x}$ and $\mathbf{y}$ as follows.

Definition 1: Let $\mathbf{x} = (\underline{x}, \overline{x})$ and $\mathbf{y} = (\underline{y}, \overline{y})$ be two intervals and let $\prec$ be a binary interval operator. The binary operation $\mathbf{x} \prec \mathbf{y}$ returns a real between 0 and 1 as

$$\mathbf{x} \prec \mathbf{y} = \begin{cases} 1 & \text{if } \underline{x} \leq \underline{y} \leq \overline{x} < \overline{y} \\ \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} & \text{if } \underline{y} \leq \underline{x} < \overline{x} \leq \overline{y} \text{ and } w(\mathbf{x}) < w(\mathbf{y}) \\ 0.5 & \text{if } w(\mathbf{x}) = w(\mathbf{y}) \text{ and } \underline{x} = \underline{y} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Since the value of $\mathbf{x} \prec \mathbf{y}$ is between 0 and 1, it can be viewed as the fuzzy membership for the statement "$\mathbf{x}$ is less than $\mathbf{y}$". This definition also works when one or both of $\mathbf{x}$ and $\mathbf{y}$ are trivial intervals. The above definition implies the following corollaries.

Corollary 1: Let $\mathbf{x}$ and $\mathbf{y}$ be two intervals, then the following holds:

1. $\mathbf{x} \prec \mathbf{y} = 0.5$ iff $m(\mathbf{x}) = m(\mathbf{y})$.

2. $\mathbf{x} \prec \mathbf{y} > 0.5$ iff $m(\mathbf{x}) < m(\mathbf{y})$.

3. $\mathbf{x} \prec \mathbf{y} < 0.5$ iff $m(\mathbf{x}) > m(\mathbf{y})$.

Proof:

We prove these three statements one by one.

1. Assume $\mathbf{x} \prec \mathbf{y} = 0.5$. If $\mathbf{x} = \mathbf{y}$, their midpoints are the same (i.e., $m(\mathbf{x}) = m(\mathbf{y})$). Otherwise, by Definition 1, $0.5 = \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} = \dfrac{\overline{y} - \overline{x}}{2r(\mathbf{y}) - 2r(\mathbf{x})}$. Hence, $\overline{y} - \overline{x} = r(\mathbf{y}) - r(\mathbf{x}) = \dfrac{\overline{y} - \underline{y}}{2} - \dfrac{\overline{x} - \underline{x}}{2}$. Therefore, $\underline{y} + \overline{y} = \underline{x} + \overline{x}$ and $m(\mathbf{x}) = m(\mathbf{y})$.

   Now, assume $m(\mathbf{x}) = m(\mathbf{y})$. If $\mathbf{x} = \mathbf{y}$, from Definition 1, $\mathbf{x} \prec \mathbf{y} = 0.5$. If $\mathbf{x} \neq \mathbf{y}$, then $\underline{y} + \overline{y} = \underline{x} + \overline{x}$. Hence, $\overline{y} - \overline{x} = \underline{x} - \underline{y}$. However, $w(\mathbf{y}) - w(\mathbf{x}) = \overline{y} - \underline{y} - (\overline{x} - \underline{x}) = (\overline{y} - \overline{x}) + (\underline{x} - \underline{y}) = 2(\overline{y} - \overline{x})$. Hence, $\mathbf{x} \prec \mathbf{y} = \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} = 0.5$.

2. Assume $\mathbf{x} \prec \mathbf{y} > 0.5$. If $\mathbf{x} \prec \mathbf{y} = 1$, then $\overline{x} < \underline{y}$. Since $m(\mathbf{x}) \leq \overline{x}$ and $\underline{y} \leq m(\mathbf{y})$, we have $m(\mathbf{x}) < m(\mathbf{y})$. If $\mathbf{x} \prec \mathbf{y} = 1^{-}$, then $\underline{x} \leq \underline{y} \leq \overline{x} < \overline{y}$. Hence, we have $\underline{x} + \overline{x} < \underline{y} + \overline{y}$. This implies $m(\mathbf{x}) < m(\mathbf{y})$. Otherwise, $\mathbf{x} \prec \mathbf{y} = \dfrac{\overline{y} - \overline{x}}{2r(\mathbf{y}) - 2r(\mathbf{x})} > 0.5$ implies $\overline{y} - \overline{x} > r(\mathbf{y}) - r(\mathbf{x})$ (i.e., $\overline{y} - \overline{x} > \dfrac{\overline{y} - \underline{y}}{2} - \dfrac{\overline{x} - \underline{x}}{2}$). Hence, $\underline{y} + \overline{y} > \underline{x} + \overline{x}$ and $m(\mathbf{x}) < m(\mathbf{y})$.

   Now assume $m(\mathbf{x}) < m(\mathbf{y})$. Then $\underline{x} + \overline{x} < \underline{y} + \overline{y}$ implies $\overline{y} - \overline{x} > r(\mathbf{y}) - r(\mathbf{x})$. If $\underline{y} \leq \underline{x} < \overline{x} \leq \overline{y}$ and $r(\mathbf{y}) > r(\mathbf{x})$, then $\mathbf{x} \prec \mathbf{y} = \dfrac{\overline{y} - \overline{x}}{2(r(\mathbf{y}) - r(\mathbf{x}))} > 0.5$. Otherwise, $\mathbf{x} \prec \mathbf{y} = 1$ or 1^{-}.

3. Assume $\mathbf{x} \prec \mathbf{y} < 0.5$. Then we have $\overline{y} - \overline{x} < r(\mathbf{y}) - r(\mathbf{x})$ (i.e., $\overline{y} - \overline{x} < \dfrac{\overline{y} - \underline{y}}{2} - \dfrac{\overline{x} - \underline{x}}{2}$). Hence, we have $\underline{x} + \overline{x} > \underline{y} + \overline{y}$. This implies $m(\mathbf{x}) > m(\mathbf{y})$.

   Now, assume $m(\mathbf{x}) > m(\mathbf{y})$. Then $\underline{x} + \overline{x} > \underline{y} + \overline{y}$ implies $\overline{y} - \overline{x} < r(\mathbf{y}) - r(\mathbf{x})$. Hence, $\mathbf{x} \prec \mathbf{y} = \dfrac{\overline{y} - \overline{x}}{2[r(\mathbf{y}) - r(\mathbf{x})]} < 0.5$.

Corollary 2: Let $\mathbf{x}$ and $\mathbf{y}$ be two intervals and $\mathbf{x} \prec \mathbf{y} \neq 1$. Then $\mathbf{x} \prec \mathbf{y} = (\mathbf{x} + \mathbf{z}) \prec (\mathbf{y} + \mathbf{z})$ for a proper interval $\mathbf{z}$.

Proof:

From the definition, if $\mathbf{x} \prec \mathbf{y} = 1^{-}$, then $\underline{x} \leq \underline{y} \leq \overline{x} < \overline{y}$. Since $\underline{z} \leq \overline{z}$, we have $\underline{x} + \underline{z} \leq \underline{y} + \underline{z} \leq \overline{x} + \overline{z} < \overline{y} + \overline{z}$. Hence, $(\mathbf{x} + \mathbf{z}) \prec (\mathbf{y} + \mathbf{z}) = 1^{-}$.

If $\mathbf{x} \prec \mathbf{y} = 0.5$, then $m(\mathbf{x}) = m(\mathbf{y})$. Hence, $m(\mathbf{x} + \mathbf{z}) = m(\mathbf{x}) + m(\mathbf{z}) = m(\mathbf{y}) + m(\mathbf{z}) = m(\mathbf{y} + \mathbf{z})$, and $(\mathbf{x} + \mathbf{z}) \prec (\mathbf{y} + \mathbf{z}) = 0.5$.

Otherwise, $(\mathbf{x} \prec \mathbf{y}) = \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})}$. Since $(\overline{y} + \overline{z}) - (\overline{x} + \overline{z}) = \overline{y} - \overline{x}$ and $w(\mathbf{y} + \mathbf{z}) - w(\mathbf{x} + \mathbf{z}) = w(\mathbf{y}) - w(\mathbf{x})$, we have $(\mathbf{x} \prec \mathbf{y}) = (\mathbf{x} + \mathbf{z}) \prec (\mathbf{y} + \mathbf{z})$.

As a dual of the above discussion, we can define a binary operator $\succ$ as the following to indicate the degree of $\mathbf{x}$ greater than $\mathbf{y}$.

Definition 2: Let $\mathbf{x}$ and $\mathbf{y}$ be two intervals and let $\succ$ be a binary interval operator that returns the fuzzy membership of the statement "$\mathbf{x}$ is greater than $\mathbf{y}$" as $(\mathbf{x} \succ \mathbf{y}) = 1 - (\mathbf{x} \prec \mathbf{y})$.

Similarly, we have the following corollary.

Corollary 3: Let $\mathbf{x}$ and $\mathbf{y}$ be two intervals. Then

1. $\mathbf{x} \succ \mathbf{y} = 0.5$ iff $m(a) = m(b)$.
2. $\mathbf{x} \succ \mathbf{y} > 0.5$ iff $m(a) > m(b)$.
3. $\mathbf{x} \succ \mathbf{y} < 0.5$ iff $m(a) < m(b)$.

3 Fuzzy Partial Order Relations for Intervals

In binary logic, a relation R on a set X is a partial order iff (a) $\forall x \in X, xRx \rightarrow$ false (inreflexive) and (b) $\forall x, y, z \in X, (xRy, yRz) \rightarrow xRz$ (transitive); then R is a partial order relation on X. To extend these concepts in fuzzy logic, we define the concepts of fuzzy inreflexibility, fuzzy transitivity, and fuzzy partial order relation as follows. We then prove that the binary operator $\prec$ is in fact a fuzzy partial order relation for intervals.

Definition 3: A fuzzy relation R on a set X is fuzzily inreflexive if $\forall x \in X, xRx = 0.5$; R is fuzzily transitive if $\forall x, y, z \in X$, if $xRy > 0.5$ and $yRz > 0.5$; then $xRz > 0.5$. If R is both fuzzily inreflexive and transitive, then R is a fuzzy partial order relation.

Theorem 1: The binary interval operators $\prec$ is a fuzzy partial order relation for intervals.

Proof:

Since $\mathbf{x} \prec \mathbf{x} = 0.5$, the operator $\prec$ is fuzzily irreflexive.

Let $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ be intervals. From Corollary 1, $\mathbf{x} \prec \mathbf{y} > 0.5$ implies $m(\mathbf{x}) < m(\mathbf{y})$, and $\mathbf{x} \prec \mathbf{y} > 0.5$ implies $m(\mathbf{y}) < m(\mathbf{z})$. The midpoints of intervals are just reals. Hence, $\mathbf{x} \prec \mathbf{y} > 0.5$ and $\mathbf{x} \prec \mathbf{y} > 0.5$ imply $m(\mathbf{x}) < m(\mathbf{z})$ and $\mathbf{x} \prec \mathbf{z} > 0.5$. Therefore, the binary operator $\prec$ is fuzzily transitive.

Hence, the binary interval operator $\prec$ is a fuzzy partial order relation for intervals.

Similarly, we can easily prove the interval operator $\succ$ forms a fuzzy partial order too.

We have now established fuzzy partial orders for intervals in terms of fuzzy membership. We complete this section with an example.

Example: For the two nested intervals $\mathbf{x} = [0, 4]$ and $\mathbf{y} = [1, 3]$, the fuzzy memberships for "$\mathbf{x}$ is less than $\mathbf{y}$" and "$\mathbf{x}$ is greater than $\mathbf{y}$" are both 0.5 since $m(\mathbf{x}) = m(\mathbf{y}) = 2$.

Letting $\mathbf{x} = [0, 3]$ and $\mathbf{y} = [0, 4]$, "$\mathbf{x}$ is less than $\mathbf{y}$" has a fuzzy membership of 1 (or 1^- to specify the weakness), whereas the fuzzy membership for "$\mathbf{x}$ is greater than $\mathbf{y}$" is zero.

For the intervals $\mathbf{x} = [1, 3]$ and $\mathbf{y} = [0, 5]$, '$\mathbf{x}$ is less than $\mathbf{y}$' has a fuzzy membership of 2/3, whereas the fuzzy membership of "$\mathbf{x}$ is greater than $\mathbf{y}$" is 1/3.

4 The $\preceq$ operator between two intervals

In studying interval valued matrix game, Collins and Hu [2] defined the $\preceq$ operator to describe the degree of an interval $\mathbf{x}$ is less than or equal to another interval $\mathbf{y}$.

Definition 4: Let $\mathbf{x}$ and $\mathbf{y}$ be two nontrivial intervals. The binary operator $\preceq$ of $\mathbf{x}$ and $\mathbf{y}$ returns the membership for "$\mathbf{x}$ is less than or equal to $\mathbf{y}$" between 0 and 1 as

$$\mathbf{x} \preceq \mathbf{y} = \begin{cases} 1 & \underline{x} \le \underline{y} \le \overline{x} < \overline{y}, \text{ or } \mathbf{x} = \mathbf{y} \\ \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} & \underline{y} < \underline{x} < \overline{x} \le \overline{y}, w(\mathbf{x}) \neq w(\mathbf{y}) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

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References

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