P1788: IEEE Standard For Interval Arithmetic Version 04.3

John Pryce and Christian Keil, Technical Editors

⚠ To the P1788 reader.

General. Passages in this color are my editorial comments: mostly asking for answers to or debate on a question; or giving my opinion; or noting changes made.

Ignore text like this: (some text)→*delete*[1]. It belongs in the full text but isn't relevant to the part currently under discussion, and it didn't seem worth rewriting the whole sentence. !

Definitions. Christian has done a lot of work on these. We believe they are now reasonably complete, and consistent with each other and the text. Only those definitions relevant to Level 1 are included in the current text.

Required and recommended elementary point functions. The lists in §5.6, 5.7 have not been voted on in their present form. I formed those in §5.6.2, 5.7.1 from the original lists of Jürgen Wolff von Gudenberg in Motion 10, much modified by subsequent discussion.

Maybe the group will accept them as they are; but if they prove controversial, we shall need separate motions to discuss them.

Provisional items. The subclauses §5.6.7 Constructors, and §5.6.8 Numeric functions of intervals Boolean functions of intervals, are purely provisional, and are not part of the present motion. I look forward to people starting discussion and submitting motions to decide these items.

Notes to version of 2012/01/23. Main changes from the previous version of 2011/12/05 are as follows.

(a) §5.6.9, Boolean functions of intervals. This is now part of the text to be voted on. I had forgotten we passed 2 motions on comparisons!

The 7 Kulisch comparisons are included as Required operations. So are isEmpty, isEntire, and areDisjoint, following the Vienna proposal.

(b) I have made the motion 21 "overlapping" function a Recommended operation (this is open to change) and given a brief explanation of how it can be used as a primitive for implementing other comparisons.

(c) I had also forgotten inner addition and subtraction (Motion 12). These are in §5.6.5 but not exactly as in Motion 12. See my justification there.

(d) A new subclause "Constants" is inserted at the start of §5.6 "Required operations". Ian McIntosh made me realize something on this is needed. This is very much a first attempt.

(e) Text strings are explicitly stated (§5.1) to be things one can talk about at Level 1.

(f) In the numeric functions of intervals, "diameter" renamed "width" as being shorter, and the function renamed from `diam` to `wid`.

[1] *CK*

2. Introduction

This introduction explains some of the alternative interpretations, and sometimes competing objectives, that influenced the design of this standard, but is not part of the standard.

2.1. Mathematical context. Interval computation is a collaboration between human programmer and machine infrastructure which, correctly done, produces mathematically proven numerical results about continuous problems—for instance, rigorous bounds on the global minimum of a function or the solution of a differential equation. It is part of the discipline of "constructive real analysis". In the long term, the results of such computations may become sufficiently trusted to be accepted as contributing to legal decisions. The machine infrastructure acts as a body of theorems on which the correctness of an interval algorithm relies, so it must be made as reliable as is practical. In its logical chain are many links—hardware, underlying floating-point system, etc.—over which this standard has no control. The standard aims to strengthen one specific link, by defining interval objects and operations that are theoretically well-founded and practical to implement. There are several mathematical bases of interval computation, which are not wholly compatible with each other. Thus it was necessary to make a choice for the purposes of this standard. The working group generally agreed that a high priority should be given to making the mathematical basis easy to grasp, easy to teach, and easy to interpret in the context of real-world applications. We believe the one we have chosen is the best match for those aims. In this theory

- Intervals are sets.
- They are subsets of the real numbers $\mathbb{R}$.
- An interval operation is defined algebraically, in contrast to topologically. The interval version of an elementary function such as $\sin x$ is essentially the natural extension to sets of the corresponding pointwise function on real numbers.

This contrasts on the one hand with *Kaucher* or *modal* interval theory, where an interval is a formal object, an ordered pair of real numbers $\underline{x}, \overline{x}$, of which those having $\underline{x} \leq \overline{x}$ are interpretable as intervals in the set sense; and on the other hand with *containment set* (cset) theory, where intervals are subsets of the extended reals $\overline{\mathbb{R}}$, and operations are defined topologically, in terms of limits. In this standard, the set $\mathbb{IR}$ of intervals is precisely the set of closed and connected (in the topological sense) subsets of $\mathbb{R}$. This includes the empty set, as well as intervals that are unbounded on one or both sides.

2.2. Specification Levels. The 754-2008 standard describes itself as layered into four Specification Levels. To manage complexity, P1788 uses a corresponding structure: level 1, of mathematical *interval theory*; level 2, the finite set of *interval datums* in terms of which finite-precision interval computation is defined; level 3, of *representations* of intervals by floating-point numbers; level 4, of *bit strings* and memory.

There is another important player: the programming language. We acknowledge the experience of the 754-2008 working group, who recognized a serious defect of the 754-1985 standard, namely that it specified individual operations but not how they should be used in expressions. Over the years, compilers made clever transformations so that it became impossible to know the precisions used and the roundings performed while evaluating an expression, or whether the compiler had even "optimized away" $(1.0 + x) - 1.0$ to become simply x.

This is also a problem for intervals. Thus the standard makes requirements and recommendations on language implementations, thereby defining the notion of a standard-conforming *implementation of intervals within a language*.

The language does not constitute a fifth level in some linear sequence; from the user's viewpoint it sits above datum level 2, alongside theory level 1, as a practical means to implement interval algorithms by manipulating level 2 entities (though most languages have influence on levels 3 and 4 also).

2.3. The Fundamental Theorem. Ramon Moore's Fundamental Theorem of Interval Arithmetic (FTIA) is central to interval computation. Roughly, it says that if f is an *explicit expression* defining a real function $f(x_1, \ldots, x_n)$, then evaluating f "in interval mode" over any interval inputs $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is guaranteed to give an enclosure of the range of f over those inputs. A version of the FTIA holds in all variants of interval theory, but with varying hypotheses and conclusions.

A standard must adhere to one theory, for if ambiguities or inconsistencies exist, a user may believe the FTIA holds in a case where it does not, with possibly serious results in applications. As stated, the FTIA is about the mathematical level. Moore's achievement was to see that "outward rounding" makes the FTIA hold also in finite precision, and to follow through the consequences. An advantage of the level structure used by the standard is that the mapping between levels 1 and 2 defines a framework where it is easily proved—indeed almost obvious—that

A conforming implementation obeys the finite-precision FTIA.

This holds also for mixed-format computation, which is supported by the standard.

2.4. Operations.

There are several interpretations of *evaluation outside an operation's domain* and *operations as relations rather than functions*. This includes classical alternative meanings of division by an interval containing zero. Consider $\boldsymbol{y} = \sqrt{\boldsymbol{x}}$ where $\boldsymbol{x} = [-1, 4]$.

1. In *optimization*, when computing lower bounds on the objective function, it is generally appropriate to return the result $\boldsymbol{y} = [0, 2]$, and ignore the fact that $\sqrt{\cdot}$ has been applied to negative elements of $\boldsymbol{x}$.
2. In applications where one must check the hypotheses of a *fixed point theorem* are satisfied (such as solving differential equations):
   (a) one may need to be sure that the function is defined and continuous on the input and, hence, throw an illegal argument exception when, as in the above case, this fails; or
   (b) one may need the result $\boldsymbol{y} = [0, 2]$, but must flag the fact that $\sqrt{\cdot}$ has been evaluated at points where it is undefined or not continuous.
3. In *constraint propagation*, the equation is often to be interpreted as: find all y such that $y^2 = x$ for some $x \in [-1, 4]$. In this case the answer is $[-2, 2]$.

The standard provides means to meet these diverse needs, hopefully without compromising clarity and efficiency. It also aims to give limited support for alternative interval theories, such as Kaucher arithmetic, for which none of the above three interpretations is appropriate.

2.5. More to be added?

⚠ I guess we need something on exception handling.

3. Notation, abbreviations, definitions

3.1. Frequently used notation and abbreviations.

754	IEEE-Std-754-2008 "IEEE Standard for Floating-Point Arithmetic".
$\mathbb{R}$	the set of real numbers.
$\overline{\mathbb{R}}$	the set of extended real numbers, $\mathbb{R} \cup \{-\infty, +\infty\}$.
$\mathbb{IR}$	the set of closed real intervals, including unbounded intervals and the empty set.
$\mathbb{F}$	generic notation for (the set of extended real numbers representable in) some floating point format.
$\mathbb{IF}$	the members of $\mathbb{IR}$ whose lower and upper bounds are in $\mathbb{F}$.
Empty	the empty set.
Entire	the whole real line.
NaI	Not an Interval.
NaN	Not a Number.
qNaN	quiet NaN.
sNaN	signaling NaN.
$x, y, \ldots$ [resp. $f, g, \ldots$]	generic notation for a numeric value [resp. numeric function].
$\boldsymbol{x}, \boldsymbol{y}, \ldots$ [resp. $\boldsymbol{f}, \boldsymbol{g}, \ldots$]	generic notation for an interval value [resp. interval function].
$\mathsf{f}, \mathsf{g}, \ldots$	generic notation for an expression, producing a function by evaluation.
Domain(f)	the domain of a point-function f.
Range($f \mid \boldsymbol{s}$)	the range of a point-function f over a set $\boldsymbol{s}$; the same as the image of $\boldsymbol{s}$ under f.

3.2. Definitions.

⚠ Definitions belonging to Levels 2 onward have been temporarily removed.

Dan Zuras notes that the Definitions subclause should be self-contained, i.e. one does not need to look outside it to understand a definition, at least in outline. Please check if this rule is flouted.

3.2.1. **arithmetic operation.** A function provided by an implementation (see Definition 3.2.9). It comes in three forms: the **point** operation, which is a mathematical real function of real variables such as $\log(x)$; one or more **interval versions**, each being an interval extension of the point operation. (; and one or more **decorated interval versions**, each being a decorated interval extension of the point operation)→*delete*[2]

Together with the interval non-arithmetic operations (§5.4.1), these form the implementation's **library**, which splits into the **point library** containing point functions and the **interval library** containing interval functions.

A **basic arithmetic operation** is one of the six functions $+$, $-$, $\times$, $\div$, fused multiply-add `fma` and square root `sqrt`.

Constants such as 3.456 and π are regarded as arithmetic operations whose number of arguments is zero. Details in §5.4.4.

3.2.2. **box.** See Definition 3.2.11.

3.2.3. **domain.** For a function with arguments taken from some set, the **domain** comprises those points in the set at which the function has a value. The domain of an arithmetic operation is part of its definition. E.g., the (point) arithmetic operation of division x/y, in this standard, has arguments (x, y) in $\mathbb{R}^2$, and its domain is the set $\{\, (x, y) \in \mathbb{R}^2 \mid y \neq 0 \,\}$. See also Definition 3.2.16.

3.2.4. **edot.** Dot product operation, that computes $\sum_{i=1}^{n} x_i y_i$.

3.2.5. **elementary function.** Synonymous with arithmetic operation.

3.2.6. **expression.** A symbolic object f that is either a symbolic variable or, recursively, of the form $\phi(\mathsf{g}_1, \ldots, \mathsf{g}_k)$, where ϕ is the name of a k-argument arithmetic or non-arithmetic operation and the g_i are expressions. It is an **arithmetic expression** if all its operations are arithmetic operations. Writing $\mathsf{f}(\mathsf{z}_1, \ldots, \mathsf{z}_n)$ makes f a **bound expression**, giving it an **argument list** comprising the variables z_i in that order, which must include all those that occur in f.

Details in §5.5.1. For the ways in which an expression defines a function, see §5.5.2, 5.5.3.

[2] *CK* 2011-10-15 Not in the motion.

3.2.7. **fma.** Fused multiply-add operation, that computes $x \times y + z$. One of the basic arithmetic operations.

3.2.8. **hull.** (Full name: **interval hull**.) When not qualified by the name of a finite-precision interval type, the hull of a subset $\boldsymbol{s}$ of $\mathbb{R}$ is the tightest interval containing $\boldsymbol{s}$.

3.2.9. **implementation.** When used without qualification, means a realization of an interval arithmetic conforming to the specification of this standard.

3.2.10. **inf-sup.** Describes a representation of an interval based on its lower and upper bounds.

3.2.11. **interval.** A closed connected set of real numbers, may be empty, bounded or unbounded. Belongs to the set of intervals $\mathbb{IR}$.

A **box** or **interval vector** is an n-dimensional interval, i. e. a tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ where the $\boldsymbol{x}_i$ are 1-dimensional intervals. Often identified with the cartesian product $\boldsymbol{x}_1 \times \ldots \times \boldsymbol{x}_n \subseteq \mathbb{R}^n$, it is empty if any of the $\boldsymbol{x}_i$ is empty. Details in §5.2.

3.2.12. **interval extension.** An interval extension of a point function f is a function $\boldsymbol{f}$ from intervals to intervals such that $f(x)$ belongs to $\boldsymbol{f}(\boldsymbol{x})$ whenever x belongs to $\boldsymbol{x}$ and $f(x)$ is defined. Details in §5.4.3.

3.2.13. **interval function, interval mapping.** A function from intervals to intervals is called an interval function if it is an interval extension of a point function, and an interval mapping otherwise. Details in §5.4.3.

3.2.14. **interval library.** See Definition 3.2.1.

3.2.15. **interval version.** See Definition 3.2.1.

3.2.16. **natural domain.** For an arithmetic expression $\mathsf{f}(\mathsf{z}_1, \ldots, \mathsf{z}_n)$, the natural domain is the set of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ where the expression defines a value for the associated point function $f(x)$. See §5.5.

3.2.17. **non-arithmetic operation.** An operation on intervals that is not an interval extension of a point operation; includes interval intersection and union.

3.2.18. **number.** Any member of the set $\mathbb{R} \cup \{-\infty, +\infty\}$ of extended reals: a **finite number** if it belongs to $\mathbb{R}$, else an **infinite number**. See §**??**

3.2.19. **point function, point operation.** A mathematical function of real variables: that is, a map f from its domain, which is a subset of $\mathbb{R}^n$, to $\mathbb{R}^m$, where $n \geq 0, m > 0$. It is **scalar** if $m = 1$. Any arithmetic expression $\mathsf{f}(\mathsf{z}_1, \ldots, \mathsf{z}_n)$ defines a (usually scalar) point function, whose domain is the natural domain of f.

3.2.20. **point library.** See Definition 3.2.1.

3.2.21. **range.** The range, $\mathrm{Range}(f \mid \boldsymbol{s})$, of a point function f over a subset $\boldsymbol{s}$ of $\mathbb{R}^n$ is the set of all values that f assumes at those points of $\boldsymbol{s}$ where it is defined, i.e. $\{\, f(x) \mid x \in \boldsymbol{s} \text{ and } x \in \mathrm{Domain}\, f \,\}$.

3.2.22. **string.** A text string, or just string, is a finite sequence of characters belonging to some alphabet. See §5.1.

3.2.23. **tightest.** Smallest in the partial order of set containment. The tightest set (unique, if it exists) with a given property is contained in every other set with that property.

Relationships between specification levels for interval arithmetic (for a given finite-precision interval type $\mathbb{T}$)		
Level 1	Number system $\mathbb{R}$. Set $\overline{\mathbb{IR}}$ of allowed intervals over $\mathbb{R}$. Principles of how $+$, $-$, $\times$, $\div$ and other arithmetic operations are extended to intervals.	Mathematical Model level.
	$\downarrow \mathbb{T}$-*interval hull* total, many-to-one[a] *identity map*$\uparrow$ total, one-to-one[b]	
Level 2	A finite subset $\mathbb{T}$ of $\overline{\mathbb{IR}}$ —the $\mathbb{T}$-interval datums— and operations on them.	Interval datum level.
	"represents"$\uparrow$ partial, many-to-one, onto[c]	
Level 3	Representation of a $\mathbb{T}$-interval, e.g. by two floating point numbers.	Representations of interval data.
	"encodes"$\uparrow$ partial, many-to-one, onto[d]	
Level 4	Encodings `0111000...`	Bit strings.

TABLE 1. Specification levels for interval arithmetic

4. Structure of the standard in levels

4.1. Specification levels overview.

The standard is structured into four levels, summarized in Table 1, that match the levels defined in the 754 standard, see 754 Table 3.1.

Level 1, in Clause 5, defines the mathematical theory underlying the standard. The entities at this level are mathematical intervals and operations on them. Conforming implementations shall implement this theory.

Level 2, in Clause 6, is the central part of the standard. Here the mathematical theory is approximated by an implementation-defined finite set of entities and operations. A level 2 entity is called a *datum* (plural "datums" in this standard, since "data" is often misleading).

An interval datum is a mathematical interval tagged by a unique *type* $\mathbb{T}$, which abstracts a particular way of representing intervals (e.g., by storing their lower and upper bounds as IEEE "single" floating point numbers). Level 2 arithmetic normally acts on intervals of a given type to produce an interval of the same type. (In addition to an ordinary (bare) interval, this level defines a *decorated* interval, comprising a bare interval and a *decoration*. Decorations implement the P1788 exception handling mechanism.)→*delete*[3]

Level 3, in Clause 7, is concerned with the representation of interval datums—usually but not necessarily in terms of floating point values. A level 3 entity is an *interval object*. The Level 3 requirements in this standard are few, and concern mappings from internal representations to external ones, such as interchange formats and I/O.

Level 4, in Clause 8, is concerned with the encoding of interval objects. A level 4 entity is a *bit string*. This standard makes no Level 4 requirements.

The arrows in Table 1 denote mappings between levels. The phrases in italics name these mappings. Each phrase "total, many-to-one", etc., labeled with a letter [a] to [d], is descriptive of the mapping and is equivalent to the corresponding labeled fact below.

a. Each mathematical interval (indeed each subset of $\mathbb{R}$) has a unique interval datum as its $\mathbb{T}$-*hull*—a minimal enclosing interval of the type $\mathbb{T}$.

b. Each interval datum is a mathematical interval.

c. Not every interval object necessarily represents an interval datum, but when it does, that datum is unique. Each interval datum has at least one representation, and may have more than one.

[3] *CK*

d. Not every interval encoding necessarily encodes an interval object, but when it does, that object is unique. Each interval object has at least one encoding and may have more than one.

4.2. Conformance requirements. ⚠ Temporarily omitted since they are almost entirely about levels 2 onward.

5. Level 1 description

In this clause, subclauses §5.1 to §5.5 describe the theory of mathematical intervals and interval functions that underlies this standard. The relation between expressions and the point or interval functions that they define is specified, since it is central to the Fundamental Theorem of Interval Arithmetic. Subclauses §5.6, 5.7 list the required and recommended *arithmetic operations* (also called elementary functions) with their mathematical specifications. (Subclause 5.8 describes, at a mathematical level, the system of *decorations* that is used among other things for exception handling in P1788.)→*delete*[4]

5.1. Non-interval Level 1 entities. in addition to intervals, the standard deals with entities of the following kinds. They may be used as inputs or outputs of operations.

– The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of **extended reals**. Following the terminology of 754 (e.g., 754§2.1.25), any member of $\overline{\mathbb{R}}$ is called a number: it is a **finite number** if it belongs to $\mathbb{R}$, else an **infinite number**.

An interval's members are finite numbers, but its bounds can be infinite. Finite or infinite numbers can be input to interval constructors, as well as output from operations, e.g., the interval width operation.

– The set of **(text) strings**, that is finite sequences of **characters** chosen from some alphabet. Since Level 1 is primarily for human communication, the standard makes no Level 1 restrictions on the alphabet used. Strings may be inputs to interval constructors, as well as inputs or outputs of read/write operations.

5.2. Intervals. The set of mathematical intervals supported by this standard is denoted $\mathbb{IR}$. It comprises the empty set (denoted $\emptyset$ or Empty) together with all nonempty closed intervals $\boldsymbol{x}$ of real numbers having the form

$$\boldsymbol{x} = [\underline{x}, \overline{x}] := \{\, x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x} \,\}, \tag{1}$$

where $\underline{x}$ and $\overline{x}$, the **bounds** of the interval, are extended-real numbers satisfying $\underline{x} \le \overline{x}$, $\underline{x} < +\infty$ and $\overline{x} > -\infty$.

[*Notes.*

– *The above definition implies $-\infty$ and $+\infty$ can be bounds of an interval, but are never members of it. For instance, $[1, +\infty]$ denotes in this document the interval $\{\, x \mid 1 \le x < +\infty \,\}$.*

– *Mathematical literature generally uses a round bracket, or reversed square bracket, to show that an endpoint is excluded from an interval, e.g. $(a, b]$ and $]a, b]$ denote $\{\, x \mid a < x \le b \,\}$. This notation is used when convenient, such as in the tables of domains and ranges of functions in §5.6, 5.7.*

– *The collection of intervals $\mathbb{IR}$ could be described more concisely as comprising all sets $\{\, x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x} \,\}$ for* arbitrary *extended-real $\underline{x}$, $\overline{x}$.*

However, this obtains Empty *in many ways, as $[\underline{x}, \overline{x}]$ for any bounds satisfying $\underline{x} > \overline{x}$, and also as $[-\infty, -\infty]$ or $[+\infty, +\infty]$. The description (1) was preferred as it makes a one-to-one mapping between valid pairs $\underline{x}, \overline{x}$ of endpoints and the nonempty intervals they specify.*

– *The notation is chosen so that $\mathbb{I}$ is an operator: for any subset A of the extended reals $\overline{\mathbb{R}}$, $\mathbb{I}A$ or $\mathbb{I}(A)$ comprises* Empty *together with all intervals (1) whose bounds belong to A. This is used at Level 2 to specify finite-precision types, e.g. $\mathbb{I}$(`binary64`) which comprises* Empty *together with all nonempty intervals whose bounds are IEEE754 double precision numbers.*

– *Excluding* Empty *is indicated by the notation $\mathbb{I}_0$, e.g. $\mathbb{I}_0\mathbb{R}$ or $\mathbb{I}_0(\mathbb{R})$, the nonempty closed intervals with (finite) real bounds, is the set used in classical interval literature from R.E. Moore onward.*

]

A **box** or **interval vector** is an n-tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ whose components $\boldsymbol{x}_i$ are intervals, that is a member of $\mathbb{IR}^n$. Usually $\boldsymbol{x}$ is identified with the cartesian product $\boldsymbol{x}_1 \times \ldots \times \boldsymbol{x}_n$ of its components, a subset of $\mathbb{R}^n$. In particular $x \in \boldsymbol{x}$, for $x \in \mathbb{R}^n$, means $x_i \in \boldsymbol{x}_i$ for all $i = 1, \ldots, n$; and $\boldsymbol{x}$ is empty if (and only if) any of its components $\boldsymbol{x}_i$ is empty.

5.3. Hull. The (interval) **hull** of an arbitrary subset $\boldsymbol{s}$ of $\mathbb{R}^n$, written hull($\boldsymbol{s}$), is the tightest member of $\mathbb{IR}^n$ that contains $\boldsymbol{s}$. (The **tightest** set with a given property is the intersection of all sets having that property, provided the intersection itself has this property.)

[4] *CK* Decorations are not part of the motion

5.4. Functions.

5.4.1. *Function terminology.* In this standard, operations are written as named functions; in a specific implementation they might be represented by operators (i.e., using some form of infix notation), or by families of type-specific functions, or by operators or functions whose names might differ from those in this standard.

The terms operation, function and mapping are broadly synonymous. The following summarizes the usage in this standard, with references in parentheses to precise definitions of terms.

- A *point function* (§5.4.2) is a mathematical real function of real variables. Otherwise, *function* is usually used with its general mathematical meaning.
- A (point) *arithmetic operation* (§5.4.2) is a mathematical real function for which an implementation provides versions in the implementation's *library* (§5.4.2).
- A *version* of a point function f means a function derived from f, such as an interval extension (§5.4.3) of it; usually applied to library functions.
- An *interval arithmetic operation* is an interval version of a point arithmetic operation (§5.4.3).
- An *interval non-arithmetic operation* is an interval-to-interval library function that is not an interval arithmetic operation (§5.4.3).
- A *constructor* is a function that creates an interval from non-interval data (§6.6.4).

5.4.2. *Point functions.* A **point function** is a (possibly partial) multivariate real function: that is, a mapping f from a subset D of $\mathbb{R}^n$ to $\mathbb{R}^m$ for some integers $n \geq 0, m > 0$. The function is called a *scalar* function if $m = 1$, otherwise a *vector* function. When not otherwise specified, scalar is assumed. The set D where f is defined is its **domain**, also written Domain f. To specify n, call f an n-variable point function, or denote values of f as

$$f(x_1, \ldots, x_n). \tag{2}$$

The **range** of f over an arbitrary subset $\boldsymbol{s}$ of $\mathbb{R}^n$ is the set

$$\text{Range}(f \,|\, \boldsymbol{s}) := \{\, f(x) \mid x \in \boldsymbol{s} \text{ and } x \in \text{Domain } f \,\}. \tag{3}$$

Thus mathematically, when evaluating a function over a set, points outside the domain are ignored (e.g., Range(sqrt $|\, [-1, 1]) = [0, 1]$).

Equivalently, for the case where f takes separate arguments $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$, each being a subset of $\mathbb{R}$, the range is written as $\text{Range}(f \,|\, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$. This is an alternative notation when $\boldsymbol{s}$ is the cartesian product of the $\boldsymbol{s}_i$.

A (point) **arithmetic operation** is a function for which an implementation provides versions in a collection of user-available operations called its **library**. This includes functions normally written in operator form (e.g., $+$, $\times$) and those normally written in function form (e.g., exp, arctan). It is not specified how an implementation provides library facilities.

5.4.3. *Interval-valued functions.* A box is an interval vector $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in \overline{\mathbb{IR}}^n$. It is usually identified with the cartesian product $\boldsymbol{x}_1 \times \ldots \times \boldsymbol{x}_n \subseteq \mathbb{R}^n$; however, the correspondence is one-to-one only when all the $\boldsymbol{x}_j$ are nonempty.

Given an n-variable scalar point function f, an **interval extension** of f is a (total) mapping $\boldsymbol{f}$ from n-dimensional boxes to intervals, that is $\boldsymbol{f} : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$, such that $f(x) \in \boldsymbol{f}(\boldsymbol{x})$ whenever $x \in \boldsymbol{x}$ and $f(x)$ is defined, equivalently

$$\boldsymbol{f}(\boldsymbol{x}) \supseteq \text{Range}(f \,|\, \boldsymbol{x})$$

for any box $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$, regarded as a subset of $\mathbb{R}^n$. The **natural interval extension** of f is defined by

$$\boldsymbol{f}(\boldsymbol{x}) := \text{hull}(\text{Range}(f \,|\, \boldsymbol{x})).$$

Equivalently, using multiple-argument notation for f, an interval extension satisfies

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \supseteq \text{Range}(f \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),$$

and the natural interval extension is defined by

$$\boldsymbol{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) := \text{hull}(\text{Range}(f \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n))$$

for any intervals $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

In some contexts it is useful for $\boldsymbol{x}$ to be a general subset of $\mathbb{R}^n$, or the $\boldsymbol{x}_i$ to be general subsets of $\mathbb{R}$; the definition is unchanged.

The natural extension is automatically defined for all interval or set arguments. (The deco- !

ration system introduced below in §5.8 gives a systematic way of diagnosing when the underlying point function has been evaluated outside its domain.)→*delete*[5]

When f is a binary operator $\bullet$ written in infix notation, this gives the usual definition of its natural interval extension as

$$\boldsymbol{x} \bullet \boldsymbol{y} = \mathrm{hull}(\{\, x \bullet y \mid x \in \boldsymbol{x},\ y \in \boldsymbol{y},\ \text{and } x \bullet y \text{ is defined} \,\}).$$

[*Example. With these definitions, the relevant natural interval extensions satisfy* $\sqrt{[-1,4]} = [0,2]$ *and* $\sqrt{[-2,-1]} = \emptyset$*; also* $\boldsymbol{x} \times [0,0] = [0,0]$ *for any nonempty* $\boldsymbol{x}$*, and* $\boldsymbol{x}/[0,0] = \emptyset$*, for any* $\boldsymbol{x}$*.*]

When f is a vector point function, a vector interval function with the same number of inputs and outputs as f is called an interval extension of f if each of its components is an interval extension of the corresponding component of f.

An interval-valued function in the library is called an **interval arithmetic operation** if it is an interval extension of a point arithmetic operation, and an **interval non-arithmetic operation** otherwise. Examples of the latter are interval intersection and union, $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \cap \boldsymbol{y}$ and $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathrm{hull}(\boldsymbol{x} \cup \boldsymbol{y})$.

5.4.4. *Constants.* A real scalar function with no arguments—a mapping $\mathbb{R}^n \to \mathbb{R}^m$ with $n = 0$ and $m = 1$—is a **real constant**. Languages may distinguish between a literal constant (e.g., the decimal value defined by the string `1.23e4`) and a named constant (e.g., π) but the difference is not relevant on Level 1 (and easily handled by outward rounding on Level 2).

From the definition, an interval extension of a real constant is any zero-argument interval function that returns an interval containing c. The *natural extension* returns the interval $[c, c]$.

5.5. Expressions and the functions they define.

5.5.1. *Expressions.* A variable is a symbolic name. A scalar **expression** in zero or more (independent) variables is a symbolic object defined to be either one of those variables or, recursively, of the form $\phi(\mathsf{g}_1, \ldots, \mathsf{g}_k)$ where ϕ is a symbolic arithmetic or non-arithmetic *operation* that takes k arguments ($k \geq 0$) and returns a single result, and the g_i are expressions. Other syntax may be used, such as traditional algebraic notation and/or program pseudo-code. An **arithmetic expression** is one in which all the operations are arithmetic operations.

A vector-valued expression is regarded, for purposes of definition in this standard, as a tuple of scalar expressions (see second example below).

[*Examples.*

– *The object* f *where*

$$\mathsf{f} = (e^x + e^{-x})/(2y)$$

is an arithmetic expression in two variables x, y *that may be written in the above form as*

$$\mathrm{div}(\mathrm{plus}(\exp(x), \exp(\mathrm{uminus}(x))), \mathrm{times}(2(), y)),$$

where uminus, plus, times, div *and* exp *name the arithmetic operations "unary minus",* $+$, $\times$, $\div$ *and exponential function. The literal constant 2 is regarded as a zero-argument arithmetic operation* $2()$*, see §5.4.4.*

The expression may be split (e.g., by a compiler) into simple assignments each involving a single operation, that may be sequenced in several ways. In the example above, one of these is

$$\begin{aligned} v_1 &= \exp(x) \\ v_2 &= \mathrm{uminus}(x) \\ v_3 &= \exp(v_2) \\ v_4 &= \mathrm{plus}(v_1, v_3) \\ v_5 &= 2() \\ v_6 &= \mathrm{times}(v_5, y) \\ f &= \mathrm{div}(v_4, v_6). \end{aligned}$$

All these forms are regarded as defining the same expression f*.*

– *A vector expression that returns the two roots of* $ax^2 + bx + c = 0$ *is regarded for definitional purposes as the two separate expressions* $(-b - \sqrt{b^2 - 4ac})/2a$ *and* $(-b + \sqrt{b^2 - 4ac})/2a$*, although in practical code it would be simplified so that common subexpressions are only evaluated once.*

– *An expression that uses the interval intersection or union operation is a non-arithmetic expression.*

[5] *CK* Not part of the motion.

]

To define functions of variables, an expression must be made into a **bound expression** by giving it a *formal argument list* with notation such as

$$f(z_1, \ldots, z_n) = \text{expression}.$$

This defines f to be the indicated expression with the formal argument list $z_1, \ldots, z_n$, which must include at least the variables that actually occur in the right-hand side. An expression without such an argument list is a **free expression**.

5.5.2. *Generic functions.* An arithmetic operation name such as $+$, or a bound arithmetic expression such as $f(x, y) = x + \sin y$, may be used to refer to different, related, functions: such operations and expressions, or the resulting functions, are called generic (or polymorphic). Whether generic function syntax can be used in program code is language-defined.

An implementation provides a number of arithmetic operation names ϕ, each representing various functions as follows.

(a) The point function of ϕ. This is unique, theoretical and generally non-computable in finite precision.
(b) Various **interval versions** of ϕ, among them in particular
  (b1) The natural interval extension of the point function. This also is unique, theoretical and generally non-computable.
  (b2) Computable interval extensions of the point function.
(c) (Decorated interval versions of ϕ, see §5.8.)→*delete*[6]

An implementation's library by definition comprises all its computable versions of required or recommended operations that it provides for any of its supported interval types, as specified in §5.6, §5.7 and in Clause 6.

An arithmetic operation or expression may also denote a floating point function; these are not defined by this standard.

5.5.3. *Point functions and interval versions of expressions.* A bound arithmetic expression $\mathsf{f} = \mathsf{f}(z_1, \ldots, z_n)$ represents various functions in the categories (a), (b1), (b2)(, (c))→*delete*[7] above.

The **point function** of (or defined by) f is a function $f : \text{Domain}\, f \subseteq \mathbb{R}^n \to \mathbb{R}$. The **natural domain** Domain f of f is the set of all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ where its value is defined according to the following rules:

– If f is the variable z_i, $f(x)$ is the number x_i. It is defined for all $x \in \mathbb{R}^n$.
– Recursively, if $\mathsf{f} = \phi(\mathsf{g}_1, \ldots, \mathsf{g}_k)$, $f(x)$ is the value of the point function of ϕ at $u = (u_1, \ldots, u_k)$ where u_i is the value of the point function of g_i at x. It is defined for those $x \in \mathbb{R}^n$ such that each of $u_1, \ldots, u_k$ is defined and the resulting u is in ϕ's domain Domain ϕ, which is part of its mathematical definition.

  In the recursive clause of this definition, ϕ can be a constant (a function with $k = 0$ arguments), so that $f(x) = \phi()$. Then, see §5.4.4, there are two cases: (a) ϕ has a real value c; then $f(x) = c$ for all $x \in \mathbb{R}^n$; (b) ϕ is the undefined function NaN, in which case $f(x)$ is not defined for any $x \in \mathbb{R}^n$.

[*Example. The specifications*

$$f(x, y) = (e^x + e^{-x})/(2y),$$
$$f(y, x) = (e^x + e^{-x})/(2y),$$
$$f(w, x, y, z) = (e^x + e^{-x})/(2y)$$

are all valid and different (they define different functions), while

$$f(y) = (e^x + e^{-x})/(2y)$$

is invalid since the argument list does not include the variable x, which occurs in f.]

[*Example. The natural domain of $f(x, y) = 1/(\sqrt{x-1} - y)$ is the set of (x, y) in the plane that do not cause either square root of a negative number or division by zero, i.e., where $x \geq 1$ and $y \neq \sqrt{x-1}$.*]

An **interval version** of $\mathsf{f}(z_1, \ldots, z_n)$ is a (total) function $f : \mathbb{IR}^n \to \mathbb{IR}$. Its value at an actual argument $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in \mathbb{IR}^n$, a box, is recursively constrained as follows:

[6] *CK* 2011-10-15 Not part of the motion
[7] *CK* 2011-10-15 Not part of the motion.

– If f is the variable z_i, the value is some interval containing $\boldsymbol{x}_i$.
– If $\mathsf{f} = \phi(\mathsf{g}_1, \ldots, \mathsf{g}_k)$, the value is some interval extension of ϕ evaluated at $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ where $\boldsymbol{u}_i$ is the value of some interval version of g_i at $\boldsymbol{x}$.

The **natural** interval version of f is the unique interval version such that when f is z_i, the value equals $\boldsymbol{x}_i$, and that uses the natural interval extension of each ϕ.

Moore's theorem states:

Theorem 5.1 (Fundamental Theorem of Interval Arithmetic, FTIA).
(i) Every interval version of an arithmetic expression $\mathsf{f}(z_1, \ldots, z_n)$ *is an interval extension of the point function defined by the expression.*
(ii) If every variable occurs at most once in f *and, for some* $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$*, all elementary operations occurring in* f *evaluate to their range, then* $f(\boldsymbol{x}) = \mathrm{Range}(f \mid \boldsymbol{x})$.

[*Note. To part (ii). Using exact arithmetic, that is, the natural interval version of* f*, it is known that an elementary operation evaluates to its range if it is continuous on its input box. (This can be checked by the decoration system (§5.8), so the conditions of (ii) are computable. The result is that in finite precision, it is often verifiable at run time that* $f(\boldsymbol{x})$ *equals* $\mathrm{Range}(f \mid \boldsymbol{x})$ *up to roundoff error.)*→delete[8]
]

[8] *CK* 2011-10-15 Not part of the motion.

5.6. Required operations.

For the interval-valued functions listed in this subclause, an implementation shall provide interval versions appropriate to its supported interval types. For constants and the forward and reverse arithmetic operations in §5.6.1, 5.6.2, 5.6.3, 5.6.4, each interval version shall be an interval extension of the corresponding point function—for a constant, that means any constant interval enclosing the point value. The required rounding behavior of these, and of the numeric functions of intervals in §5.6.8, is detailed in §6.5, 6.6. The dot product in §5.6.10 is exceptional: it is a point function of point values, whose required rounding behavior is detailed in §**??**.

The names of operations in this standard, as well as symbols used for operations (e.g., for the comparisons in §5.6.9), may not correspond to those that any particular language would use.

5.6.1. *Constants.*

⚠ NEW. Ian McIntosh's comments made me aware we need requirements about denoting (a) exact numbers (possibly non-computable in a given precision), (b) intervals with exact number endpoints (also possibly non-computable), (c) finite precision hulls of such intervals (computable). This is my first attempt. It needs cleaning up by a language expert.

An implementation shall provide means to denote exact point (numerical) constants including $\pm\infty$, and exact intervals whose endpoints are such constants.

The denotable point constants shall include all values expressible as floating constants in the C language, in decimal or hexadecimal form, and of arbitrary finite precision. This standard writes such denotations as strings using C syntax and calls them **literal constants**, e.g. the decimal literal `"1.234e-5"` denoting 1.234×10^{-5}, or the hexadecimal literal `"3.Fp+10"` denoting $3\frac{15}{16} \times 2^{10}$. An implementation may also provide **named constants**, e.g. `"Pi"` representing π.

Denotations of intervals are written as strings following the syntax for input/output of intervals in §6.11: e.g., `"[1.234e5,Inf]"` or `"Entire"`.

Where needed, the map from a denotation to its value is written `eval`. For instance, `eval("[1.234e-5,Inf]")` equals $[1.234 \times 10^{-5}, +\infty]$. [*Note. This map is a notational device, not an operation to be implemented, but for denotations of intervals it is equivalent to the Level 1 operation* `text2interval`.]

How an implementation provides constants is language-defined or implementation-defined as well as locale-dependent. E.g., it may represent numerical constants in the style `'1.234e5'` of a lexical type "symbol" that is distinct from "string". It may categorize constants otherwise than as "literal" or "named".

An implementation may provide other ways to denote exact intervals, e.g., a denotation using midpoint and radius as part of support for a mid-rad interval type.

5.6.2. *Forward-mode elementary functions.*

Table 2 on page 18 lists required arithmetic operations. The term *operation* includes functions normally written in function notation $f(x, y, \ldots)$, as well as those normally written in unary or binary operator notation, $\bullet x$ or $x \bullet y$.

[*Note. The list includes all general-computational operations in 754*§*5.4 except* **convertFromInt**, *and some recommended functions in 754*§*9.2.*]

Proving the correctness of interval computations relies on the Fundamental Theorem of Interval Arithmetic, which in turn relies on the relation between a point-function and its interval versions. Thus the domain of each point-function and its value at each point of the domain are specified to aid a rigorous implementation. This is mostly straightforward but needs care for functions with discontinuities, such as pow() and atan2().

⚠ We probably should have a version of interval **division** x/y that returns two intervals so that it doesn't lose information when 0 is an interior point of y. Several solutions exist, e.g. Kulisch's; Kearfott's, which is similar; and Vienna proposal's "division with gap". A motion please!

5.6.3. *Interval case expressions and case function.*

Functions are often defined by conditionals: $f(x)$ equals $g(x)$ if some condition on x holds, and $h(x)$ otherwise. To handle interval extensions of such functions in a way that automatically conforms to the Fundamental Theorem of Interval Arithmetic, the ternary function `case`(c, g, h) is provided. To simplify defining its interval extension, the argument c specifying the condition is

TABLE 2. Required forward elementary functions.
Normal mathematical notation is used to include or exclude an interval endpoint, e.g., $(-\pi, \pi]$ denotes $\{\, x \in \mathbb{R} \mid -\pi < x \leq \pi \,\}$.

Name	Definition	Point function domain	Point function range	Note
`add`(x, y)	$x + y$	$\mathbb{R}^2$	$\mathbb{R}$	
`sub`(x, y)	$x - y$	$\mathbb{R}^2$	$\mathbb{R}$	
`mul`(x, y)	xy	$\mathbb{R}^2$	$\mathbb{R}$	
`div`(x, y)	x/y	$\mathbb{R}^2 \setminus \{y = 0\}$	$\mathbb{R}$	a
`recip`(x)	$1/x$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$	
`sqrt`(x)	$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$	
`hypot`(x, y)	$\sqrt{x^2 + y^2}$	$\mathbb{R}^2$	$[0, \infty)$	
`case`(c, g, h)		See §5.6.3.		
`sqr`(x)	x^2	$\mathbb{R}$	$[0, \infty)$	
`pown`(x, p)	x^p, $p \in \mathbb{Z}$	$\begin{cases}\mathbb{R} \text{ if } p \geq 0 \\ \mathbb{R}\setminus\{0\} \text{ if } p < 0\end{cases}$	$\begin{cases}\mathbb{R} \text{ if } p > 0 \text{ odd} \\ [0, \infty) \text{ if } p > 0 \text{ even} \\ \{1\} \text{ if } p = 0 \\ \mathbb{R}\setminus\{0\} \text{ if } p < 0 \text{ odd} \\ (0, \infty) \text{ if } p < 0 \text{ even}\end{cases}$	b
`pow`(x, y)	x^y	$\{x{>}0\} \cup \{x{=}0, y{>}0\}$	$[0, \infty)$	a, c
`exp,exp2,exp10`(x)	b^x	$\mathbb{R}$	$(0, \infty)$	d
`log,log2,log10`(x)	$\log_b x$	$(0, \infty)$	$\mathbb{R}$	d
`sin`(x)		$\mathbb{R}$	$[-1, 1]$	
`cos`(x)		$\mathbb{R}$	$[-1, 1]$	
`tan`(x)		$\mathbb{R}\setminus\{(k + \frac{1}{2})\pi \mid k \in \mathbb{Z}\}$	$\mathbb{R}$	
`asin`(x)		$[-1, 1]$	$[-\pi/2, \pi/2]$	e
`acos`(x)		$[-1, 1]$	$[0, \pi]$	e
`atan`(x)		$\mathbb{R}$	$(-\pi/2, \pi/2)$	e
`atan2`(y, x)		$\mathbb{R}^2 \setminus \{\langle 0, 0\rangle\}$	$(-\pi, \pi]$	e, f, g
`sinh`(x)		$\mathbb{R}$	$\mathbb{R}$	
`cosh`(x)		$\mathbb{R}$	$[1, \infty)$	
`tanh`(x)		$\mathbb{R}$	$(-1, 1)$	
`asinh`(x)		$\mathbb{R}$	$\mathbb{R}$	
`acosh`(x)		$[1, \infty)$	$[0, \infty)$	
`atanh`(x)		$(-1, 1)$	$\mathbb{R}$	
`sign`(x)		$\mathbb{R}$	$\{-1, 0, 1\}$	h
`ceil`(x)		$\mathbb{R}$	$\mathbb{Z}$	i
`floor`(x)		$\mathbb{R}$	$\mathbb{Z}$	i
`round`(x)		$\mathbb{R}$	$\mathbb{Z}$	i
`trunc`(x)		$\mathbb{R}$	$\mathbb{Z}$	i
`abs`(x)	$\lvert x\rvert$	$\mathbb{R}$	$[0, \infty)$	
`min`$(x_1, \ldots, x_k)$		$\mathbb{R}^k$ for $k = 2, 3, \ldots$	$\mathbb{R}$	j
`max`$(x_1, \ldots, x_k)$		$\mathbb{R}^k$ for $k = 2, 3, \ldots$	$\mathbb{R}$	j

Notes to Table 2

a. In describing the domain, notation such as $\{y = 0\}$ is short for $\{\, (x, y) \in \mathbb{R}^2 \mid y = 0 \,\}$, etc.
b. Regarded as a family of functions parameterized by the integer argument p.
c. Defined as $e^{y \ln x}$ for real $x > 0$ and all real y, and 0 for $x = 0$ and $y > 0$, else undefined.
d. $b = e, 2$ or 10, respectively.
e. The ranges shown are the mathematical range of the point function. To ensure containment, an interval result may include values just outside the mathematical range.
f. atan2(y, x) is the principal value of the argument (polar angle) of (x, y) in the plane.
g. To avoid confusion with notation for open intervals, in this table coordinates in $\mathbb{R}^2$ are delimited by angle brackets $\langle\ \rangle$.
h. `sign`(x) is -1 if $x < 0$; 0 if $x = 0$; and 1 if $x > 0$.
i. `ceil`(x) is the smallest integer $\geq x$. `floor`(x) is the largest integer $\leq x$. `round`(x) is the nearest integer to x, with ties rounded according to the language default. `trunc`(x) is the nearest integer to x in the direction of zero. (As defined in the C standard §7.12.9.)
j. Smallest, or largest, of its real arguments. A family of functions parameterized by the arity k.

real (instead of boolean), and the condition means $c < 0$ by definition. That is,

$$\mathtt{case}(c, g, h) = \begin{cases} g & \text{if } c < 0, \\ h & \text{otherwise.} \end{cases}$$

Its natural interval extension is

$$\mathtt{case}(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{h}) = \begin{cases} \emptyset & \text{if } \boldsymbol{c} \text{ is empty} \\ \boldsymbol{g} & \text{elseif } \overline{c} < 0 \\ \boldsymbol{h} & \text{elseif } \underline{c} \geq 0 \\ \mathrm{hull}(\boldsymbol{g} \cup \boldsymbol{h}) & \text{otherwise.} \end{cases} \tag{4}$$

for any intervals $\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{h}$, where $\boldsymbol{c} = [\underline{c}, \overline{c}]$ when nonempty.

The function f above may be encoded as $f(x) = \mathtt{case}\big(c(x), g(x), h(x)\big)$. Then, if $\boldsymbol{c}$, $\boldsymbol{g}$, $\boldsymbol{h}$ are interval extensions of c, g and h, the function

$$\boldsymbol{f}(\boldsymbol{x}) = \mathtt{case}\big(\boldsymbol{c}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x})\big) \tag{5}$$

is automatically an interval extension of f.

[*Notes.*

1. *This method is less awkward than using interval comparisons as a mechanism for handling such functions. However, the resulting interval function is usually not the tightest extension of the corresponding point function. E.g., the absolute value $f(x) = |x|$ may be defined by*

   $$f(x) = \mathtt{case}(x, -x, x).$$

   Then it is easy to see that formula (5), applied to a nonempty $\boldsymbol{x} = [\underline{x}, \overline{x}]$, gives the exact range $\{\, |x| \mid x \in \boldsymbol{x} \,\}$ when $\overline{x} < 0$ or $0 \leq \underline{x}$, but the poor enclosure $(-\boldsymbol{x}) \cup \boldsymbol{x}$ when $\underline{x} < 0 \leq \overline{x}$.
2. *$\mathtt{case}(c, g, h)$ is equivalent to the C expression $(c < 0\,?\,g : h)$.*
3. *Compound conditions may be expressed using the `max` and `min` operations: e.g., a real function $f(x, y)$ that equals $\sin(xy)$ in the positive quadrant of the plane, and zero elsewhere, may be written*

   $$f(x, y) = \mathtt{case}(\min(x, y), 0, \sin(xy)),$$

   since $\min(x, y) < 0$ is equivalent to ($x < 0$ or $y < 0$).

]

5.6.4. *Reverse-mode elementary functions.*

Constraint-satisfaction algorithms use the functions in this subclause for iteratively tightening an enclosure of a solution to a system of equations.

Given a unary arithmetic operation ϕ, a **reverse interval extension** of ϕ is a binary interval function $\phi\mathrm{Rev}$ such that

$$\phi\mathrm{Rev}(\boldsymbol{c}, \boldsymbol{x}) \supseteq \{\, x \in \boldsymbol{x} \mid \phi(x) \text{ is defined and in } \boldsymbol{c} \,\}, \tag{6}$$

for any intervals $\boldsymbol{z}, \boldsymbol{x}$.

Similarly, a binary arithmetic operation $\bullet$ has two forms of reverse interval extension, which are ternary interval functions $\bullet\mathrm{Rev}_1$ and $\bullet\mathrm{Rev}_2$ such that

$$\bullet\mathrm{Rev}_1(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{x}) \supseteq \{\, x \in \boldsymbol{x} \mid b \in \boldsymbol{b} \text{ exists such that } x \bullet b \text{ is defined and in } \boldsymbol{c} \,\}, \tag{7}$$

$$\bullet\mathrm{Rev}_2(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{x}) \supseteq \{\, x \in \boldsymbol{x} \mid a \in \boldsymbol{a} \text{ exists such that } a \bullet x \text{ is defined and in } \boldsymbol{c} \,\}. \tag{8}$$

If $\bullet$ is commutative then $\bullet\mathrm{Rev}_1$ and $\bullet\mathrm{Rev}_2$ agree and may be implemented simply as $\bullet\mathrm{Rev}$.

In each of (6, 7, 8), the unique **natural reverse interval extension** is the one whose value is the interval hull of the right-hand side. Clearly, any reverse interval extension encloses this hull.

The last argument $\boldsymbol{x}$ in each of (6, 7, 8) is optional, with default $\boldsymbol{x} = \mathbb{R}$ if absent.

[*Note. The argument $\boldsymbol{x}$ can be thought of as giving prior knowledge about the range of values taken by a point-variable x, which is then sharpened by applying the reverse function: see the example below.*]

Reverse operations shall be provided as in Table 3. Note $\mathtt{pownRev}(x, p)$ is regarded as a family of unary functions parametrized by p.

[*Example.*

– *Consider the function $sqr(x) = x^2$. Evaluating $sqr\mathrm{Rev}([1, 4])$ answers the question: given that $1 \leq x^2 \leq 4$, what interval can we restrict x to? Using the natural reverse extension, we have*

$$sqr\mathrm{Rev}([1, 4]) = \mathsf{hull}\{\, x \in \mathbb{R} \mid x^2 \in [1, 4] \,\} = \mathsf{hull}([-2, -1] \cup [1, 2]) = [-2, 2].$$

TABLE 3. Required reverse elementary functions.

From unary functions	From binary functions
sqrRev$(\boldsymbol{c},\boldsymbol{x})$	mulRev$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{x})$
invRev$(\boldsymbol{c},\boldsymbol{x})$	divRev1$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{x})$
absRev$(\boldsymbol{c},\boldsymbol{x})$	divRev2$(\boldsymbol{a},\boldsymbol{c},\boldsymbol{x})$
pownRev$(\boldsymbol{c},\boldsymbol{x},p)$	powRev1$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{x})$
sinRev$(\boldsymbol{c},\boldsymbol{x})$	powRev2$(\boldsymbol{a},\boldsymbol{c},\boldsymbol{x})$
cosRev$(\boldsymbol{c},\boldsymbol{x})$	atan2Rev1$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{x})$
tanRev$(\boldsymbol{c},\boldsymbol{x})$	atan2Rev2$(\boldsymbol{a},\boldsymbol{c},\boldsymbol{x})$
coshRev$(\boldsymbol{c},\boldsymbol{x})$	

– *If we can add the prior knowledge that $x \in \boldsymbol{x} = [0, 1.2]$, then using the optional second argument gives the tighter enclosure*

$$sqr\mathrm{Rev}([1,4],[0,1.2]) = \mathsf{hull}\{\, x \in [0,1.2] \mid x^2 \in [1,4] \,\} = \mathsf{hull}\big([0,1.2] \cap ([-2,-1] \cup [1,2])\big) = [1,1.2].$$

– *One might think it suffices to apply the operation without the optional argument and intersect the result with $\boldsymbol{x}$. This is less effective because "hull" and "intersect" do not commute. E.,g., in the above, this method evaluates*

$$sqr\mathrm{Rev}([1,4]) \cap \boldsymbol{x} = [-2,2] \cap [0,1.2] = [0,1.2],$$

so no tightening of the enclosure $\boldsymbol{x}$ is obtained.

]

5.6.5. *Inner addition and subtraction.*

⚠ I have made this only apply to bounded intervals, since it really seems hard to frame a specification for unbounded ones that translates unambiguously to the finite precision (Level 2) situation.

Also I have deviated from the Motion 12 spec, by making the operations defined only when width$(\boldsymbol{x}) \geq$ width$(\boldsymbol{y})$. IMO it is actively harmful in applications to make it always defined. This is for the same reasons that it is harmful to replace $\sqrt{x}$ by the everywhere defined $\sqrt{|x|}$: an application MUST test for definedness, and making it always defined leads to un-noticed wrong results from code that forgets to test.

Inner subtraction solves the problem: Recover interval $\boldsymbol{z}$ from intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, given that one knows $\boldsymbol{x}$ was obtained as the sum $\boldsymbol{z} + \boldsymbol{y}$.

[*Example. In some applications one has a list of intervals $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and needs to form each interval $\boldsymbol{s}_k$ which is the sum of all the $\boldsymbol{a}_i$ except $\boldsymbol{a}_k$, that is $\boldsymbol{s}_k = \sum_{i=1,\, i\neq k}^{n} \boldsymbol{a}_i$, for $k = 1, \ldots, n$.*

Evaluating all these sums independently costs $O(n^2)$ work. However, if one forms the sum $\boldsymbol{s}$ of all *the $\boldsymbol{a}_i$, one can obtain each $\boldsymbol{s}_k$ from $\boldsymbol{s}$ and $\boldsymbol{a}_k$ by inner subtraction. This method only costs $O(n)$ work.*

This example illustrates that in finite precision, computing $\boldsymbol{x}$ (as a sum of terms) typically incurs at least one roundoff error, and may incur many. Thus one should think of $\boldsymbol{x}$ as an enclosure of an unknown true sum $\boldsymbol{x}_0$, whereas y is "exact". The computed $\boldsymbol{z}$ is thus an enclosure of an unknown true $\boldsymbol{z}_0$ such that $\boldsymbol{z}_0 + \boldsymbol{y} = \boldsymbol{x}_0$.]

There is an operation innerPlus$(\boldsymbol{x},\boldsymbol{y})$, equivalent to innerMinus$(\boldsymbol{x},-\boldsymbol{y})$ and therefore not specified separately.

There is an operation innerMinus$(\boldsymbol{x},\boldsymbol{y})$ that returns for any two bounded intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ the tightest interval $\boldsymbol{z}$ such that

$$\boldsymbol{z} + \boldsymbol{y} \supseteq \boldsymbol{x} \tag{9}$$

if such a $\boldsymbol{z}$ exists, and is undefined otherwise.

This specification leads to the following Level 1 algorithm. If $\boldsymbol{x} = \emptyset$ then $\boldsymbol{z} = \emptyset$. If $\boldsymbol{x} \neq \emptyset$ and $\boldsymbol{y} = \emptyset$ then $\boldsymbol{z}$ is undefined. If $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = [\underline{y}, \overline{y}]$ are both nonempty and bounded, define $\underline{z} = \underline{x} - \underline{y}$ and $\overline{z} = \overline{x} - \overline{y}$. Then $\boldsymbol{z}$ is defined to be $[\underline{z}, \overline{z}]$ if $\underline{z} \leq \overline{z}$ (equivalently if width$(\boldsymbol{x}) \geq$ width$(\boldsymbol{y})$), and is undefined otherwise. Otherwise (if either interval is unbounded), $\boldsymbol{z}$ is undefined.

[*Note. Because of the cancellative nature of these operations, care is needed in finite precision to determine whether the result is defined or not. More details are given at Level 3 in §7.5.*

E.g., ⚠ (Maybe this should move to Level 3) *consider inf-sup intervals using 3 decimal digit floating point arithmetic. Let $\boldsymbol{x} = [.0001, 1]$ and $\boldsymbol{y} = [-1, -.0002]$. Thus $\boldsymbol{x}$ is slightly the wider, so*

$z_1 = \texttt{innerMinus}(\boldsymbol{x}, \boldsymbol{y})$ is defined and is [1.0001, 1.0002]; while $z_2 = \texttt{innerMinus}(\boldsymbol{y}, \boldsymbol{x})$ is not defined. However, one cannot discriminate these cases using naive 3 digit arithmetic. Comparing $\texttt{width}(\boldsymbol{x})$ with $\texttt{width}(\boldsymbol{y})$ gives the wrong result, because both are computed (rounding upward) as 1.01, suggesting z_2 is defined. Computing the bounds of z_2, namely $[(-1.00 - .0001), (-.0002 - 1.00)]$ (with outward rounding), also gives the wrong result, namely $[-1.01, -1.00]$, again suggesting z_2 is defined. Only real or simulated higher precision is guaranteed to give the correct decision in all cases.]

5.6.6. *Non-arithmetic operations.*

The following operations shall be provided, the arguments and result being intervals.

Name	Definition
$\texttt{intersection}(\boldsymbol{x}, \boldsymbol{y})$	$\boldsymbol{x} \cap \boldsymbol{y}$
$\texttt{convexHull}(\boldsymbol{x}, \boldsymbol{y})$	interval hull of $\boldsymbol{x} \cup \boldsymbol{y}$

5.6.7. *Constructors.*

The following operations shall be provided, that create an interval from non-interval data.

The operation $\texttt{nums2interval}(l, u)$, where l and u are extended-real values, returns the set $\{\, x \in \mathbb{R} \mid l \le x \le u \,\}$. If (see §5.2) the conditions $l \le u$, $l < +\infty$ and $u > -\infty$ hold, this set is the nonempty interval $[l, u]$ and the operation is said to *succeed*. Otherwise the operation is said to *fail*, and returns Empty.

Success and failure are used in specifying how decorations are set by the corresponding constructors of decorated intervals at Level 2.

The operation $\texttt{num2interval}(x)$ is equivalent to $\texttt{nums2interval}(x, x)$. It fails if and only if x is infinite.

The operation $\texttt{text2interval}(t)$ succeeds and returns the interval denoted by the text string t, if t denotes an interval. Otherwise, it fails and returns Empty.

[*Note. Since Level 1 is mainly for human-human communication, any understandable t is acceptable, e.g.* `"[3.1,4.2]"` *or* `"[2π,∞]"`*. Rules for the strings t accepted at an implementation level are given in the Level 2 Subclause 6.11 on I/O and may optionally be followed.*]

5.6.8. *Numeric functions of intervals.*

The operations in Table 4 shall be provided, the argument being an interval and the result a number, which for some of the operations may be infinite.

5.6.9. *Boolean functions of intervals.*

The following operations shall be provided, which return a boolean (1 = true, 0 = false) result.

There is a function $\texttt{isEmpty}(\boldsymbol{x})$, which returns 1 if $\boldsymbol{x}$ is the empty set, 0 otherwise. There is a function $\texttt{isEntire}(\boldsymbol{x})$, which returns 1 if $\boldsymbol{x}$ is the whole line, 0 otherwise.

There are eight comparison relations, which take two interval inputs and return a boolean result. For *nonempty* intervals $\boldsymbol{a}$ and $\boldsymbol{b}$, these are defined in Table 5. If in any of $\boldsymbol{a} = \boldsymbol{b}$, $\boldsymbol{a} \subseteq \boldsymbol{b}$ or $\boldsymbol{a} \le \boldsymbol{b}$ both operands are the empty set $\emptyset$, the result is true. If either operand is $\emptyset$ in $\boldsymbol{a} \not\cap \boldsymbol{b}$, the result is true. If $\boldsymbol{a} = \emptyset$ in $\boldsymbol{a} \subseteq \boldsymbol{b}$ or $\boldsymbol{a} \subset \boldsymbol{b}$, the result is true. Otherwise the result is false if in any of the first seven relations either operand is $\emptyset$.

[*Note. All these relations, except $\boldsymbol{a} \not\cap \boldsymbol{b}$, are transitive. The first three are reflexive. Relation $\boldsymbol{a} \prec \boldsymbol{b}$, though its symbol suggests it is reflexive, is not so.*]

5.6.10. *Dot product function.*

The function $\texttt{dotProduct}(x, y)$ is provided, where x and y are real vectors with the same number of elements. It returns $\sum_i x_i y_i$.

This is a point-function; an interval extension of it is not *required*. The most significant part of its specification concerns its rounding properties in finite precision, which are specified in §6.6.

TABLE 4. Required numeric functions of an interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$.
Note `inf` can return $-\infty$; each of `sup`, `wid`, `rad` and `mag` can return $+\infty$.

Name	Definition
`inf`($\boldsymbol{x}$)	lower bound of $\boldsymbol{x}$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty
`sup`($\boldsymbol{x}$)	upper bound of $\boldsymbol{x}$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty
`mid`($\boldsymbol{x}$)	midpoint $(\underline{x} + \overline{x})/2$ if $\boldsymbol{x}$ is nonempty bounded undefined if $\boldsymbol{x}$ is empty or unbounded
`wid`($\boldsymbol{x}$)	width $\overline{x} - \underline{x}$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty
`rad`($\boldsymbol{x}$)	radius $(\overline{x} - \underline{x})/2$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty
`midRad`($\boldsymbol{x}$)	returns the pair (mid($\boldsymbol{x}$), rad($\boldsymbol{x}$)) undefined if either member is undefined
`mag`($\boldsymbol{x}$)	magnitude $\sup\{\, \lvert x\rvert \mid x \in \boldsymbol{x} \,\}$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty
`mig`($\boldsymbol{x}$)	mignitude $\inf\{\, \lvert x\rvert \mid x \in \boldsymbol{x} \,\}$ if $\boldsymbol{x}$ is nonempty undefined if $\boldsymbol{x}$ is empty

⚠ I have gone for simplicity. Also I have presumptuously backed my own view that at Level 1 it is more consistent with math conventions for a function to be simply "undefined" outside its domain, than for it to return a special value. At Level 2 this translates into returning a value such as NaN.

TABLE 5. Comparisons for nonempty intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$. Recall that $\underline{a}, \underline{b}$ may be $-\infty$ and $\overline{a}, \overline{b}$ may be $+\infty$.

Name	Symbol	Definition	Description
`isEqual`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} = \boldsymbol{b}$	$\underline{a} = \underline{b} \wedge \overline{a} = \overline{b}$	$\boldsymbol{a}$ equals $\boldsymbol{b}$
`containedIn`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \subseteq \boldsymbol{b}$	$\underline{b} \le \underline{a} \wedge \overline{a} \le \overline{b}$	$\boldsymbol{a}$ is a subset of $\boldsymbol{b}$
`lessEqual`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \le \boldsymbol{b}$	$\underline{a} \le \underline{b} \wedge \overline{a} \le \overline{b}$	$\boldsymbol{a}$ is less than or equal to $\boldsymbol{b}$
`precedesEqual`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \preceq \boldsymbol{b}$	$\overline{a} \le \underline{b}$	$\boldsymbol{a}$ precedes or touches $\boldsymbol{b}$
`containedInInterior`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \subset \boldsymbol{b}$	$\underline{b} < \underline{a} \wedge \overline{a} < \overline{b}$	$\boldsymbol{a}$ is interior to $\boldsymbol{b}$
`less`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} < \boldsymbol{b}$	$\underline{a} < \underline{b} \wedge \overline{a} < \overline{b}$	$\boldsymbol{a}$ is strictly less than $\boldsymbol{b}$
`precedes`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \prec \boldsymbol{b}$	$\overline{a} < \underline{b}$	$\boldsymbol{a}$ precedes $\boldsymbol{b}$
`areDisjoint`($\boldsymbol{a}, \boldsymbol{b}$)	$\boldsymbol{a} \not\cap \boldsymbol{b}$	$\overline{a} < \underline{b} \vee \overline{b} < \underline{a}$	$\boldsymbol{a}$ and $\boldsymbol{b}$ are disjoint

5.7. Recommended operations (informative).
Language standards should define interval versions of some or all functions in this subclause, and some or all supported types, as is most appropriate to the language.

5.7.1. *Forward-mode elementary functions.*
The list of recommended functions is in Table 6. Each interval version provided is required to be an interval extension of the point function.

5.7.2. *Extended interval comparisons.*
The **interval overlapping operation** `overlap`($\boldsymbol{a}, \boldsymbol{b}$), also written $\boldsymbol{a} \circledcirc \boldsymbol{b}$, arises from the work of J.F. Allen [**?**] on temporal logic. It may be used as an infrastructure for other interval comparisons. If implemented, it should also be available at user level; how this is done is implementation-defined or language-defined.

Allen identified 13 states of a pair $(\boldsymbol{a}, \boldsymbol{b})$ of nonempty intervals, which are ways in which they can be related with respect to the usual order $a < b$ of the reals. Together with three states for when either interval is empty, these define the 16 possible values of `overlap`($\boldsymbol{a}, \boldsymbol{b}$).

To describe the states for nonempty intervals of positive width, it is useful to think of $\boldsymbol{b} = [\underline{b}, \overline{b}]$ (with $\underline{b} < \overline{b}$) as fixed, while $\boldsymbol{a} = [\underline{a}, \overline{a}]$ (with $\underline{a} < \overline{a}$) starts far to its left and moves to the right. Its

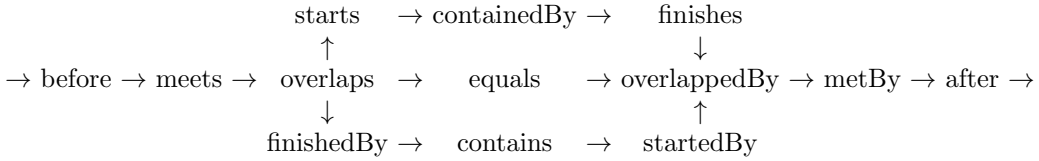
TABLE 6. Recommended elementary functions.
Normal mathematical notation is used to include or exclude an interval endpoint, e.g., $(-1, 1]$ denotes $\{\, x \in \mathbb{R} \mid -1 < x \leq 1 \,\}$.

Name	Definition	Point function domain	Point function range	Note
`rootn`(x, q)	real $\sqrt[q]{x}$, $q \in \mathbb{Z} \setminus \{0\}$	$\begin{cases}\mathbb{R} \text{ if } q > 0 \text{ odd}\\ [0,\infty) \text{ if } q > 0 \text{ even}\\ \mathbb{R}\backslash\{0\} \text{ if } q < 0 \text{ odd}\\ (0,\infty) \text{ if } q < 0 \text{ even}\end{cases}$	same as domain	a
`powr`(x, p, q)	real $x^{p/q}$	$\begin{cases}\mathbb{R} \text{ if } r \geq 0,\ s \text{ odd}\\ [0,\infty) \text{ if } r > 0,\ s \text{ even}\\ \mathbb{R}\backslash\{0\} \text{ if } r < 0,\ s \text{ odd}\\ (0,\infty) \text{ if } r < 0,\ s \text{ even}\end{cases}$	$\begin{cases}\mathbb{R} \text{ if } r > 0,\ rs \text{ odd}\\ [0,\infty) \text{ if } r > 0,\ rs \text{ even}\\ \{1\} \text{ if } r = 0\\ \mathbb{R}\backslash\{0\} \text{ if } r < 0,\ rs \text{ odd}\\ (0,\infty) \text{ if } r < 0,\ rs \text{ even}\end{cases}$	a
	$p \in \mathbb{Z}$, $q \in \mathbb{Z} \setminus \{0\}$;	r/s is p/q in lowest terms, see Note c.		
`expm1`(x) `exp2m1`(x) `exp10m1`(x)	$b^x - 1$	$\mathbb{R}$	$(-1,\infty)$	b, d
`logp1`(x) `log2p1`(x) `log10p1`(x)	$\log_b(x+1)$	$(-1,\infty)$	$\mathbb{R}$	b, d
`compoundm1`(x, y)	$(1+x)^y - 1$	$\{x{>}{-1}\} \cup \{x{=}{-1}, y{>}0\}$	$[0,\infty)$	d, e
`rSqrt`(x)	$1/\sqrt{x}$	$(0,\infty)$	$(0,\infty)$	
`sinPi`(x)	$\sin(\pi x)$	$\mathbb{R}$	$[-1,1]$	f
`cosPi`(x)	$\cos(\pi x)$	$\mathbb{R}$	$[-1,1]$	f
`tanPi`(x)	$\tan(\pi x)$	$\mathbb{R}\backslash\{\, k + \frac{1}{2} \mid k \in \mathbb{Z} \,\}$	$\mathbb{R}$	f
`asinPi`(x)	$\arcsin(x)/\pi$	$[-1,1]$	$[-1/2, 1/2]$	f
`acosPi`(x)	$\arccos(x)/\pi$	$[-1,1]$	$[0,1]$	f
`atanPi`(x)	$\arctan(x)/\pi$	$\mathbb{R}$	$(-1/2, 1/2)$	f
`atan2Pi`(y, x)	$\operatorname{atan2}(y,x)/\pi$	$\mathbb{R}^2 \setminus \{\langle 0,0\rangle\}$	$(-1,1]$	f, g

Notes to Table 6

a. Regarded as a family of functions parameterized by the integer arguments q, or r and s.
b. $b = e, 2$ or 10, respectively.
c. Defined as $(\sqrt[s]{x})^r$ where r/s is the reduced fraction equal to p/q, where integers r and s have no common factor, $s > 0$. E.g., `powr`$(x, -2, -6) = x^{(-2)/(-6)}$ reduces to $\sqrt[3]{x}$ and `powr`$(x, 4, -6) = x^{4/(-6)}$ to $(\sqrt[3]{x})^{-2}$.
d. Mathematically unnecessary, but included to let implementations give better numerical behavior for small values of the arguments.
e. In describing domains, notation such as $\{y = 0\}$ is short for $\{\, (x, y) \in \mathbb{R}^2 \mid y = 0 \,\}$, and so on.
f. These functions avoid a loss of accuracy due to π being irrational, cf. Table 2, note e.
g. To avoid confusion with notation for open intervals, in this table coordinates in $\mathbb{R}^2$ are delimited by angle brackets $\langle\ \rangle$.

endpoints move continuously with strictly positive velocity. Then, depending on the relative sizes of $\boldsymbol{a}$ and $\boldsymbol{b}$, the value of $\boldsymbol{a} \circledcirc \boldsymbol{b}$ follows a path from left to right through the graph below, whose nodes represent Allen's 13 states.



For instance "$\boldsymbol{a}$ overlaps $\boldsymbol{b}$"—equivalently $\boldsymbol{a} \circledcirc \boldsymbol{b}$ has the value `overlaps`—is the case $\underline{a} < \underline{b} < \overline{a} < \overline{b}$.

The three extra values are: `bothEmpty` when $\boldsymbol{a} = \boldsymbol{b} = \emptyset$, else `firstEmpty` when $\boldsymbol{a} = \emptyset$, `secondEmpty` when $\boldsymbol{b} = \emptyset$.

Table 7 shows the 16 states, with the 13 "nonempty" states specified (a) in terms of set membership using quantifiers and (b) in terms of the endpoints $\underline{a}, \overline{a}, \underline{b}, \overline{b}$, and also (c) shown diagrammatically.

The set and endpoint specifications remove some ambiguities of the diagram view when one interval shrinks to a single point that coincides with an endpoint of the other. Such a case is allocated to `equal` when all four endpoints coincide; else to `starts`, `finishes`, `finishedBy` or `startedBy` as appropriate; never to `meets` or `metBy`.

[*Note. The 16 state values can be encoded in four bits. However, if they are then translated into patterns P in a 16-bit word, having one position equal to 1 and the rest zero, one can easily implement interval comparisons by using bit-masks.*

For instance, suppose we make the states s in Table 7's order correspond to the 16 bits in the word, left-to-right, so $s =$ `bothEmpty` *maps to $P(s) =$* `1000000000000000`*, $s =$* `firstEmpty` *maps to $P(s) =$* `0100000000000000` *and so on. Consider the relation* `areDisjoint`$(\boldsymbol{a}, \boldsymbol{b})$*. This is true if and only if one or both of $\boldsymbol{a}$ or $\boldsymbol{b}$ is empty, or $\boldsymbol{a}$ is "before" $\boldsymbol{b}$, or $\boldsymbol{a}$ is "after" $\boldsymbol{b}$. That is, iff the logical "and" of $P(s)$ with the mask* `disjointMask = 1111000000000001` *is not identically zero.*

This scheme can be efficiently implemented in hardware, see for instance M. Nehmeier, S. Siegel and J. Wolff von Gudenberg [**?**]*. All the required comparisons in this standard can be implemented in this way, as can be, e.g., the "possibly" and "certainly" comparisons of Sun's interval Fortran. Thus the overlap operation is a primitive from which it is simple to derive all interval comparisons commonly found in the literature.*]

5.7.3. *Slope functions.*

The functions in Table 8 are the commonest ones needed to efficiently implement improved range enclosures via *first- and second-order slope* algorithms. They are analytic at $x = 0$ after filling in the removable singularity there, where each has the value 1.

TABLE 7. The 16 states of interval overlapping situations for intervals $\boldsymbol{a}, \boldsymbol{b}$. Notation $\forall_a$ means "for all a in $\boldsymbol{a}$", and so on. Phrases within a cell are joined by "and", e.g. `starts` is specified by $(\underline{a} = \underline{b} \wedge \overline{a} < \overline{b})$.

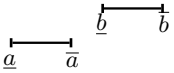
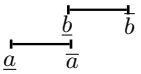
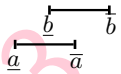
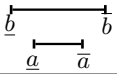
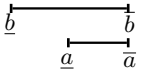
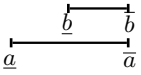
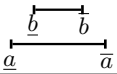
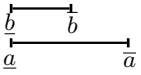
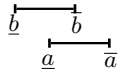
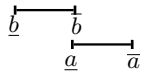
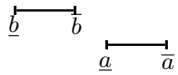
State $\boldsymbol{a} \circledcirc \boldsymbol{b}$ is	Set specification	Endpoint specification	Diagram
States with either interval empty			
`bothEmpty`	$\boldsymbol{a} = \emptyset \wedge \boldsymbol{b} = \emptyset$		
`firstEmpty`	$\boldsymbol{a} = \emptyset \wedge \boldsymbol{b} \neq \emptyset$		
`secondEmpty`	$\boldsymbol{a} \neq \emptyset \wedge \boldsymbol{b} = \emptyset$		
States with both intervals nonempty			
`before`	$\forall_a \forall_b\ a < b$	$\overline{a} < \underline{b}$	
`meets`	$\forall_a \forall_b\ a \leq b$ $\exists_a \forall_b\ a < b$ $\exists_a \exists_b\ a = b$	$\underline{a} < \overline{a}$ $\overline{a} = \underline{b}$ $\underline{b} < \overline{b}$	
`overlaps`	$\exists_a \forall_b\ a < b$ $\exists_b \forall_a\ a < b$ $\exists_a \exists_b\ b < a$	$\underline{a} < \underline{b}$ $\underline{b} < \overline{a}$ $\overline{a} < \overline{b}$	
`starts`	$\exists_a \forall_b\ a \leq b$ $\exists_b \forall_a\ b \leq a$ $\exists_b \forall_a\ a < b$	$\underline{a} = \underline{b}$ $\overline{a} < \overline{b}$	
`containedBy`	$\exists_b \forall_a\ b < a$ $\exists_b \forall_a\ a < b$	$\underline{b} < \underline{a}$ $\overline{a} < \overline{b}$	
`finishes`	$\exists_b \forall_a\ b < a$ $\exists_a \forall_b\ b \leq a$ $\exists_b \forall_a\ a \leq b$	$\underline{b} < \underline{a}$ $\overline{a} = \overline{b}$	
`equal`	$\forall_a \exists_b\ a = b$ $\forall_b \exists_a\ b = a$	$\underline{a} = \underline{b}$ $\overline{a} = \overline{b}$	
`finishedBy`	$\exists_a \forall_b\ a < b$ $\exists_b \forall_a\ a \leq b$ $\exists_a \forall_b\ b \leq a$	$\underline{a} < \underline{b}$ $\overline{b} = \overline{a}$	
`contains`	$\exists_a \forall_b\ a < b$ $\exists_a \forall_b\ b < a$	$\underline{a} < \underline{b}$ $\overline{b} < \overline{a}$	
`startedBy`	$\exists_b \forall_a\ b \leq a$ $\exists_a \forall_b\ a \leq b$ $\exists_a \forall_b\ b < a$	$\underline{b} = \underline{a}$ $\overline{b} < \overline{a}$	
`overlappedBy`	$\exists_b \forall_a\ b < a$ $\exists_a \forall_b\ b < a$ $\exists_b \exists_a\ a < b$	$\underline{b} < \underline{a}$ $\underline{a} < \overline{b}$ $\overline{b} < \overline{a}$	
`metBy`	$\forall_b \forall_a\ b \leq a$ $\exists_b \exists_a\ b = a$ $\exists_b \forall_a\ b < a$	$\underline{b} < \overline{b}$ $\overline{b} = \underline{a}$ $\underline{a} < \overline{a}$	
`after`	$\forall_b \forall_a\ b < a$	$\overline{b} < \underline{a}$	

TABLE 8. Recommended slope functions.

Name	Definition	Point function domain	Point function range	Note
`expSlope1`(x)	$\frac{1}{x}(e^x - 1)$	$\mathbb{R}$	$(0, \infty)$	
`expSlope2`(x)	$\frac{2}{x^2}(e^x - 1 - x)$	$\mathbb{R}$	$(0, \infty)$	
`logSlope1`(x)	$\frac{2}{x^2}(\log(1+x) - x)$	$\mathbb{R}$	$(0, \infty)$	
`logSlope2`(x)	$\frac{3}{x^3}(\log(1+x) - x + \frac{x^2}{2})$	$\mathbb{R}$	$(0, \infty)$	
`cosSlope2`(x)	$-\frac{2}{x^2}(\cos x - 1)$	$\mathbb{R}$	$[0, 1]$	
`sinSlope3`(x)	$-\frac{6}{x^3}(\sin x - x)$	$\mathbb{R}$	$(0, 1]$	
`asinSlope3`(x)	$\frac{6}{x^3}(\arcsin x - x)$	$[-1, 1]$	$[1, 3\pi - 6]$	
`atanSlope3`(x)	$-\frac{3}{x^3}(\arctan x - x)$	$\mathbb{R}$	$(0, 1]$	
`coshSlope2`(x)	$\frac{2}{x^2}(\cosh x - 1)$	$\mathbb{R}$	$[1, \infty)$	
`sinhSlope3`(x)	$\frac{3}{x^3}(\sinh x - x)$	$\mathbb{R}$	$[\frac{1}{2}, \infty)$	