Modal Intervals: Brief Description

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Abstract

This note provides a brief description of modal intervals as described in [1].

Traditional interval computations: reminder. Let us assume that a quantity z depends on quantities $x = (x_1, \ldots, x_n)$, and that we know the exact form of this dependence, i.e., we know a continuous function $z = f(x) = f(x_1, \ldots, x_n)$. In practice, we often do not know the exact values of the quantities x_i, we only know the intervals $X_i = [\underline{x}_i, \overline{x}_i]$ that contain these values.

These intervals may come from *measurements:* when the measurement result is $\widetilde{x}_i$ and we know the upper bound Δ_i on (absolute value of) the measurement error $\Delta x_i \stackrel{\rm def}{=} \widetilde{x}_i - x_i$, this means that the actual (unknown) value x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. These intervals can also come from *manufacturing tolerances*, when we recommend the value $\widetilde{x}_i$ of the corresponding quantity but allow deviations $\pm\Delta_i$ from this recommended value. In this case also, the resulting the resulting quantity x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

In both cases, the only information that we have about z is that z belongs to the interval

$$Z = \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\} = \left[\min_{x \in X} f(x), \max_{x \in X} f(x)\right],$$

where we denoted $X \stackrel{\rm def}{=} X_1 \times \ldots X_n$. This interval Z is called the *result of applying the function* f to the intervals $X_1, \ldots, X_n$ and denoted by $f(X_1, \ldots, X_n)$.

In many practical situations, it is desirable to make the interval Z as narrow as possible. For example, z may be the direction of the airplane flight, and we want to maintain this direction as accurately as possible. In the above setting, if we want to decrease the width Z, we have to decrease the width of the original intervals – e.g., measure the values x_i more accurately, or impose stricter tolerances on the manufacturing process.

Logical reformulation of the traditional interval computation. First, we need to make sure that for all possible combinations of $x_i \in X_i$, the value

$z = f(x_1, \ldots, x_n)$ is contained in the interval Z. In other words, we want to make sure that

$$\forall x_1 \in X_1 \ldots \forall x_n \in X_n \, \exists z \in Z \, (z = f(x_1, \ldots, x_n)).$$

Second, we need to make sure that Z is the narrowest interval with this property. These two requirements guarantee that Z is equal to the above range: $Z = f(X)$.

Beyond the main problem of (traditional) interval computations – possibility of controlled variables: formulation of the problem. In the traditional approach, we have no control over the values of the input variables x_i, we only know that these values belong to the corresponding intervals X_i. In practice, often, the desired value $z = f(x, u)$ depends not only the variables $x = (x_1, \ldots, x_n)$ over which we have no control, it also depends on the additional variables $u = (u_1, \ldots, u_m)$ that we *can* control. Specifically, for each of these additional variables u_j, there is a range U_j, and we can set up any value within this range. We can use these additional variables to narrow down the range $Z = [\underline{z}, \overline{z}]$ of the values z that can be achieved.

In precise terms, we want to select an interval $Z = [\underline{z}, \overline{z}]$ for which, for each combination $x \in X$, there exists a control u that would lead to the value $f(x, u) \in Z$. Among all such intervals Z, we want to select the one which is the narrowest. In other words, we want to make sure that

$$\forall x \in X \, \exists u \in U \, (f(x, u) \in Z),$$

i.e., that

$$\forall x \in X \, \exists u \in U \, \exists z \in Z \, (z = f(x, u)),$$

and that Z is the narrowest interval with this property.

How can we find such an interval Z?

Possibility of controlled variables: towards a solution to the problem. For each $x \in X$, the set of all possible values $f(x, u)$ forms an interval

$$F(x) \stackrel{\text{def}}{=} \left[\min_{u \in U} f(x, u), \max_{u \in U} f(x, u)\right].$$

The existence of a control u for which one of these values is from the interval Z is equivalent to requiring that that the intervals $F(x)$ and Z have a common point. One can easily check that the two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ have a common point if and only if $\underline{a} \le \overline{b}$ and $\underline{b} \le \overline{a}$. For intervals $F(x)$ and Z, this means that we must have

$$\min_{u \in U} f(x, u) \le \overline{z} \text{ and } \underline{z} \le \max_{u \in U} f(x, u).$$

These two inequalities much hold for every $x \in X$. For $\overline{z}$, this means that the value $\overline{z}$ must be larger than or equal to $\min\limits_{u \in U} f(x, u)$ for all $x \in X$. This is equivalent to requiring that $\overline{z}$ is larger than or equal to the largest of these values, i.e., that

$$\overline{z} \geq \max_{x\in X}\min_{u\in U} f(x,u).$$

Similarly, the requirement that $\underline{z}$ must be smaller than or equal to $\max\limits_{u\in U} f(x,u)$ for all $x \in X$ is equivalent to requiring that $\underline{z}$ is smaller than or equal to the smallest of these values, i.e., that

$$\underline{z} \leq \min_{x\in X}\max_{u\in U} f(x,u).$$

Among all the intervals that satisfy these two inequalities, we need to find the narrowest. It turns out that the selection of the narrowest interval depends on the relation between the two bounds. If

$$\min_{x\in X}\max_{u\in U} f(x,u) \leq \max_{x\in X}\min_{u\in U} f(x,u),$$

then the narrowest interval is when $\overline{z}$ is equal to its lower bound and $\underline{z}$ is equal to its upper bound, i.e., when

$$Z = [\underline{z}, \overline{z}] = \left[\min_{x\in X}\max_{u\in U} f(x,u), \max_{x\in X}\min_{u\in U} f(x,u)\right].$$

On the other hand, if the opposite inequality is satisfied, i.e., if

$$\min_{x\in X}\max_{u\in U} f(x,u) > \max_{x\in X}\min_{u\in U} f(x,u),$$

then we can have intervals Z with the desired property which have width 0: namely, for any value z between these two bounds, i.e., for any value z from the interval

$$Z = \left[\max_{x\in X}\min_{u\in U} f(x,u), \min_{x\in X}\max_{u\in U} f(x,u)\right],$$

the one-point interval $Z' = [z, z]$ satisfies the desired property.

Thus, we arrive at the following solution.

Case of controlled variables: solution. Once we have a function $f(x,u)$ and the ranges X and U, we compute the two values

$$z^- = \min_{x\in X}\max_{u\in U} f(x,u) \text{ and } z^+ = \max_{x\in X}\min_{u\in U} f(x,u).$$

If $z^- \leq z^+$, then the interval $Z = [z^-, z^+]$ is the narrowest interval for which

$$\forall x \in X\, \exists z \in Z\, \exists u \in U\, (z = f(x,u)).$$

If $z^- > z^+$, then we have many such narrowest intervals – namely, every interval $[z, z]$ for $z \in [z^+, z^-]$ is a one. This can be described as follows:

$$\forall x \in X\, \forall z \in Z\, \exists u \in U\, (z = f(x,u)).$$

Comment. The above solution is presented in [1], where the pair consisting of the values z^- and z^+ is called an f^*-extension of the original function $f(x, u)$.

Reformulation in terms of modal intervals. In [1], logical terms are used to distinguish between intervals X_i over which we have no control and intervals U_j in which we can select whichever value $u_i \in U_i$ we choose. To guarantee that the value z of the desired quantity is within the given range, we need to make sure that this property holds *for all* possible values $x_i \in X_i$, while for the controlled intervals, it is sufficient to require that *there exist* values $u_j \in U_j$ that make this property true. To emphasize this distinction, the authors of [1] treat each interval as a pair of the interval itself and of the corresponding quantifier:

- a traditional interval X_i is considered as a pair $\langle X_i, \forall \rangle$, while
- a controlled interval is considered as a pair $\langle U_j, \exists \rangle$.

Such pairs are called *modal intervals*.

In these terms, the condition

$$\forall x_1 \in X_1 \ldots \forall x_n \in X_n \, \exists u_1 \in U_1 \ldots \exists u_m \in U_m \; \exists z \in Z \, (z = f(x, u))$$

can be reformulated as

$$Q x_1 \in X_1 \ldots Q x_n \in X_n \, Q u_1 \in U_1 \ldots Q u_m \in U_m \; \exists z \in Z \, (z = f(x, u)),$$

where Q is the quantifier attached to the corresponding interval. For the case when all the intervals are traditional (non-controlled), we get the usual expression for the range. Because of this example, we can treat the resulting interval Z as the range defined over modal intervals:

$$Z = f(\langle X_1, \forall \rangle, \ldots, \langle X_n, \forall \rangle, \langle U_1, \exists \rangle, \ldots, \langle U_m, \exists \rangle).$$

The difference between the cases $z^- \le z^+$ and $z^- > z^+$ translates, as we have seen, into the difference between $\exists z \in Z$ and $\forall z \in Z$ in the corresponding formulas. So, the authors of [1] say that when $z^- \le z^+$, the range is the usual interval $\langle Z, \forall \rangle$, while for $z^- > z^+$, the range is the interval $\langle Z, \exists \rangle$.

Relation to Kaucher intervals. The above example shows that the difference between the two types of intervals can also be represented as the difference between the usual intervals, for which $z^- \le z^+$, and the "new" intervals for which $z^- > z^+$. It is therefore reasonable to represent these "new intervals" as $[z^-, z^+]$.

For example, the interval $Z = \langle [2, 4], \forall \rangle$ is represented as a usual interval $[2, 4]$, while an interval $\langle [2, 4], \exists \rangle$ is represented as $[4, 2]$. Such intervals have been previously introduced by Kaucher.

This connection with Kaucher intervals is not accidental: indeed, for arithmetic operations $f(x, u)$, the f^*-extensions coincide with the operations of Kaucher arithmetic.

References

[1] E. Gardeñes et al., "Modal intervals", *Reliable Computing*, 2001, Vol. 7, pp. 77–111.