Motion 10: Elementary Functions

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1 Motion

P1788 shall provide a fixed set of mandatory elementary functions listed in Table 1 and a set of recommended elementary functions listed in Table 2. Table 2 is implementation defined. The tables list the (P1788) name of the function, its mathematical definition with domain and range as well as the required accuracy model.

We assume that an accuracy model will be defined for the standard. In this motion we use the model of the Vienna Proposal, specifying the modes "tightest", "accurate", and "valid".

In accordance with motion 5 (arithmetic operators) we define
$f(\mathbf{x}) = \Diamond\{f(x)|x \in \mathbf{x}$ and $f(x)$ is defined.$\}$
Here $\Diamond$ means the interval hull.

Some of the functions will also be provided in "reverse mode", but that shall be subject of another motion.

Evaluation of functions outside their domain shall be handled with the common exception handling scheme. Currently that means decorated intervals proposed as motion 8 that is still under discussion.

name	definition	domain	range	accuracy
sqr	x^2	$\mathbb{R}$	$[0, \infty)$	tightest
pown	x^n	$\mathbb{R} \times \mathbb{Z}$	$\mathbb{R}$	accurate
pow	x^y	$[0, \infty) \times \mathbb{R}$	$[0, \infty)$	
sqrt		$[0, \infty)$	$[0, \infty)$	tightest
exp		$\mathbb{R}$	$[0, \infty)$	tightest
log	$\ln x$	$(0, \infty)$	$\mathbb{R}$	tightest
log10	$\log_{10} x$	$(0, \infty)$	$\mathbb{R}$	tightest
log2	$\log_2 x$	$(0, \infty)$	$\mathbb{R}$	tightest
sin		$\mathbb{R}$	$[-1, 1]$	accurate
cos		$\mathbb{R}$	$[-1, 1]$	accurate
tan		$\mathbb{R} - \{(2k+1)\pi/2 \vert k \in \mathbb{Z}\}$	$\mathbb{R}$	accurate
asin		$[-1, 1]$	$[-\pi/2, \pi/2]$	accurate
acos		$[-1, 1]$	$[0, \pi]$	accurate
atan		$\mathbb{R}$	$(-\pi/2, \pi/2)$	accurate
atan2		$\mathbb{R} \times \mathbb{R}$	$(-\pi/2, \pi/2)$	accurate
sinh		$\mathbb{R}$	$\mathbb{R}$	accurate
cosh		$\mathbb{R}$	$[1, \infty)$	accurate
tanh		$\mathbb{R}$	$[-1, 1]$	accurate
asinh		$\mathbb{R}$	$\mathbb{R}$	accurate
acosh		$[1, \infty)$	$[0, \infty)$	accurate
atanh		$[-1, 1]$	$\mathbb{R}$	accurate

Table 1: Required elementary functions

name	definition	domain	range	accuracy
abs		$\mathbb{R}$	$[0, \infty)$	tightest
sign3		$\mathbb{R}$	$\{-1, 0, 1\}$	tightest
ceil		$\mathbb{R}$	$\mathbb{Z}$	tightest
floor		$\mathbb{R}$	$\mathbb{Z}$	tightest
root	$x^{(1/q)}$	$[0, \infty)$	$[0, \infty)$	accurate
powr	$x^{p/q}$	$[0, \infty) \times \mathbb{Z} \times \mathbb{Z}$	$[0, \infty)$	accurate
rSqrt	$1/\sqrt{x}$	$[0, \infty) \times \mathbb{Z} \times \mathbb{Z}$	$[0, \infty)$	accurate
hypot	$\sqrt{x^2 + y^2}$	$\mathbb{R} \times \mathbb{R}$	$[0, \infty)$	accurate
compound	$(1+x)^n$	$[-1, \infty) \times \mathbb{Z}$	$[0, \infty)$	accurate
exp2	$(e^x - 1 - x)/x^2$	$\mathbb{R}$	$[0, \infty)$	accurate
cos2	$(\cos(x) - 1)/x^2$	$\mathbb{R}$	$(-\infty, 0]$	accurate
sin2	$(\sin x - x)/x^2$	$\mathbb{R}$	$[-\infty, 0]$	accurate
cosh2	$(\cosh(x) - 1)/x^2$	$\mathbb{R}$	$[[0, \infty)$	accurate
sinh2	$(\sinh x - x)/x^2$	$\mathbb{R}$	$[0, \infty)$	accurate
exp1	$(e^x - 1 - x)/x$	$\mathbb{R}$	$\mathbb{R}$	accurate
expm1	$e^x - 1$	$\mathbb{R}$	$[-1, \infty)$	tightest
logp1	$ln(x+1)$	$[-1, \infty)$	$\mathbb{R}$	tightest

Table 2: Recommended elementary functions

2 Rationale

The naming of the functions follows the 754 standard, whereas the slightly changed selection is that of the Vienna Proposal.

We divide the elementary functions into 2 sets. Mandatory functions comprise the usual set of math standard functions provided in languages and libraries for scientific applications. Namely power, exponential, and logarithmic functions as well as the trig and hyperbolic functions together with their inverses. Recommended functions include convenience functions such as $expm1(x) = e^x - 1$ as well as simple utility functions, like floor. Unary functions like $-x$ or $1/x$ are considered as operators, while power and atan2 are binary functions.

Table 3 shows a possible implementation of the interval square root function for bare intervals using a correctly rounded implementaion of the floating point sqrt function.

X	sqrt(X)
$x_1 < x_2 < 0$	$\emptyset$
$x_1 < 0 \leq x_2$	$[0, \Delta sqrt(x_2)]$
$x_1 < 0, x_2 = \infty$	$[0, \infty]$
$0 <= x_1 \leq x_2$	$[\nabla sqrt(x_1), \Delta sqrt(x_2)]$
$0 <= x_1, x_2 = \infty$	$[\nabla sqrt(x_1), \infty]$

Table 3: sqrt for bare intervals

The decoration trits for definedness and boundedness are touched by the calculation of square root. Just as an example we show how the definedness propagates in the last 3 colums of Table 4. Please note that details concerning the decoration are not topic of this motion, but of motion 8.

X	sqrt(X)	X.deco,def		
		no	maybe	yes
$x_1 < x_2 < 0$	$\emptyset$	no	no	no
$x_1 < 0 \leq x_2$	$[0, \Delta sqrt(x_2)]$	no	maybe	maybe
$x_1 < 0, x_2 = \infty$	$[0, \infty]$	no	maybe	maybe
$0 <= x_1 \leq x_2$	$[\nabla sqrt(x_1), \Delta sqrt(x_2)]$	no	maybe	yes
$0 <= x_1, x_2 = \infty$	$[\nabla sqrt(x_1), \infty]$	no	maybe	yes

Table 4: sqrt for decorated intervals