Interval Angles and the Fortran ATAN2 Intrinsic Function

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Abstract

Conventions are considered that prevent incorrect results in both point and interval analysis of angles. A convention is proposed with which to resolve the inherent ambiguity in directed angles when computing an interval enclosure of the Fortran `ATAN2` intrinsic function.

1 Introduction

From James and James [1]:

> "There are two commonly used signed measures of directed angles. If a circle is drawn with unit radius and center at the vertex of a directed angle, then a radian measure of the angle is the length of an arc that extends counterclockwise along the circle from the initial side to the terminal side of the angle, or the negative of the length of an arc that extends clockwise along the circle from the initial side to the terminal side. The arc may wrap around the circle any number of times. For example, if an angle has radian measure $\frac{1}{2}\pi$, it also has radian measure $\frac{1}{2}\pi + 2\pi$, $\frac{1}{2}\pi + 4\pi$, etc., or $\frac{1}{2}\pi - 2\pi$, $\frac{1}{2}\pi - 4\pi$, etc. A rotation angle consists of a directed angle *and* a signed measure of the angle. The angle is a positive angle or a negative angle according as the measure is positive or negative. Equal rotation angles are rotation angles that have the same measure. Usually, angle means rotation angle. A rotation angle can be thought of as being a directed angle together with a description of how the angle is formed by rotating a ray from an initial position (on the initial side) to a terminal position (on the terminal side)."

2 Point Angles

Before considering conventions needed to deal with interval angles, it is helpful to review the conventions commonly used for points.

2.1 Radian measures of rotation and directed angles

Radian measures of rotation angles are real numbers and can be mathematically manipulated in any meaningful way. The directed angle corresponding to a given rotation angle does not contain the number of full rotations a ray takes between its initial and the terminal position. A directed angle may or may not retain information about the direction of rotation of the ray used to create its corresponding rotation angle. Respectively, let ω and θ denote the radian measures of a rotation angle and a possible corresponding directed angle. Then ω and θ are related as follows:

$$\omega = \theta + 2k\pi,$$

where k is an integer function of ω. If the direction of rotation of the ray used to define ω is preserved in θ, ω and θ can be related as follows:

$$\theta = \omega - \operatorname{trunc}\left(\frac{\omega}{2\pi}\right) 2\pi, \tag{1}$$

in which case

$$k = \operatorname{trunc}\left(\frac{\omega}{2\pi}\right), \tag{2}$$

where

$$\operatorname{trunc}(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x \geq 0, \text{ and} \\ \lceil x \rceil, & \text{if } x < 0. \end{cases} \tag{3}$$

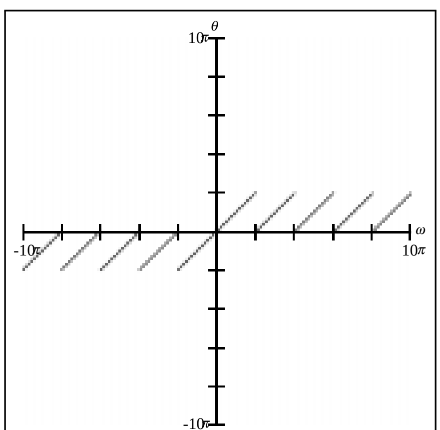

Figure 1: $\theta = \omega - \operatorname{trunc}\left(\frac{\omega}{2\pi}\right) 2\pi$

If the direction of rotation is not preserved in θ, ω and θ can be related in a variety of ways, including the following three:

1.
$$\theta = \omega \bmod 2\pi = \omega - \left\lfloor \frac{\omega}{2\pi} \right\rfloor 2\pi, \tag{4a}$$

   in which case

$$k = \left\lfloor \frac{\omega}{2\pi} \right\rfloor, \text{ or} \tag{4b}$$

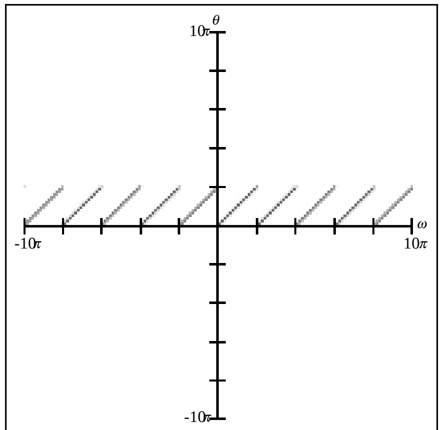

Figure 2: $\theta = \omega - \left\lfloor \frac{\omega}{2\pi} \right\rfloor 2\pi$

2.

$$\theta = -\left(-\omega \bmod 2\pi\right) = \omega - \left\lceil \frac{\omega}{2\pi} \right\rceil 2\pi, \tag{5a}$$

in which case

$$k = \left\lceil \frac{\omega}{2\pi} \right\rceil, \text{ or} \tag{5b}$$

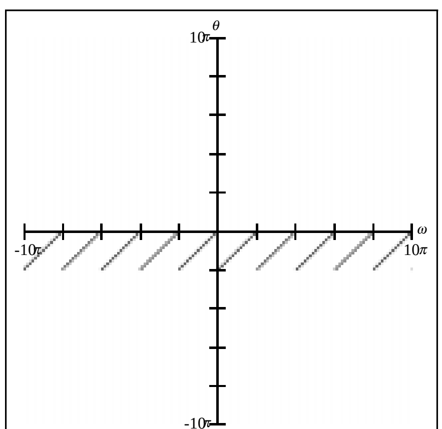

Figure 3: $\theta = \omega - \left\lceil \frac{\omega}{2\pi} \right\rceil 2\pi$

3.

$$\theta = (\omega \pm \pi) \bmod 2\pi - \pi = \omega - \left\lfloor \frac{\omega + \pi}{2\pi} \right\rfloor 2\pi \tag{6a}$$

$$= -\left(-\left(\omega \pm \pi\right) \bmod 2\pi\right) + \pi = \omega - \left\lceil \frac{\omega - \pi}{2\pi} \right\rceil 2\pi \tag{6b}$$

in which case

$$k = \left\lfloor \frac{\omega + \pi}{2\pi} \right\rfloor \tag{6c}$$

$$= \left\lceil \frac{\omega - \pi}{2\pi} \right\rceil . \tag{6d}$$

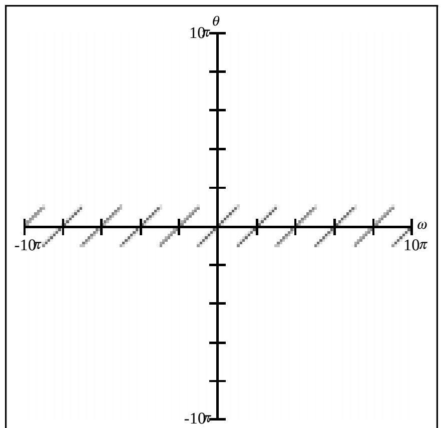

Figure 4: $\theta = \omega - \left\lfloor \frac{\omega+\pi}{2\pi} \right\rfloor 2\pi$

The choice of how to relate rotation and direction angles is a matter of convention and convenience.

Directed angles are inherently ambiguous. Depending on the value of the rotation angle and the chosen relation between rotation and direction angles, different rotation angles can have the same or different directed angles. For example, the two rotation angles with radian measures $\frac{1}{2}\pi$ and $\frac{5}{2}\pi$ can have the same or different directed angles. Even the rotation angle $\frac{1}{2}\pi$ can have the directed angle $\frac{1}{2}\pi$ or $\frac{-3}{2}\pi$. The rotation angle $\frac{1}{2}\pi$ is greater than the rotation angle $\frac{-1}{2}\pi$, but some definitions of the corresponding direction angles can erroneously lead to the opposite conclusion. For example, the rotation angle $\frac{-1}{2}\pi$ can have the direction angle $\frac{3}{2}\pi$. Therefore, when comparing radian measures of directed angles, care must be taken to add or subtract the appropriate multiple of 2π to guarantee that the difference between two directed angles, $\theta_1 - \theta_2$, is the same as the corresponding difference between their respective rotation angles.

2.2 Trigonometric function conventions

Trigonometric functions of rotation angles contain even worse ambiguity than directed angles because they are not one-to-one mappings within a single 2π rotation of a ray. To unambiguously define inverse trigonometric functions, conventions have been established for the principle radian measure intervals that define the domain of each inverse trigonometric function. For example, the principle radian measure intervals for the cosine and sine are $[0, \pi]$ and

$\left[\frac{-1}{2}\pi, \frac{1}{2}\pi\right]$, respectively. These conventions have been followed in the Fortran standard.

2.3 Fortran ATAN2(Y,X) function

Given either a directed angle's sine or cosine, it is not possible to recover the angle's radian measure. However radian measure recovery is possible from the combination of the angle's sine and cosine, or numbers proportional thereto. Given $y = h \sin\theta$ and $x = h\cos\theta$, the value of θ can be recovered, modulo 2π. For example, if $y = 0$ and $x > 0$, $\theta = 2k\pi$ for some integer value of k. If $y = 0$ and $x < 0$, $\theta = (2k-1)\pi$ for some integer value of k. Similarly, if $x = 0$, and $y > 0$, $\theta = \left(2k + \frac{1}{2}\right)\pi$, but if $y < 0$, $\theta = \left(2k - \frac{1}{2}\right)\pi$, for some integer value of k. By convention, the range of the Fortran `ATAN2(Y,X)` function is defined to be $[-\pi, \pi]$. This corresponds to the mapping of rotation angles to direction angles in (6a) and number of rotations, k, in (6c). Algorithm developers must explicitly save and restore the number of rotations, k, needed to reconstruct a rotation angle, ω, from a direction angle, θ. Thus, the Fortran `ATAN2(Y,X)` function returns

$$\theta = ((\omega + \pi) \bmod 2\pi) - \pi, \tag{7}$$

given

$$y = h \sin\omega, \tag{8a}$$

$$x = h\cos\omega, \text{ and} \tag{8b}$$

$$0 < h < \infty. \tag{8c}$$

If $h = 0$ or ∞, θ is indeterminate.

3 Interval angles

An interval angle is an interval with endpoints that are radian measures. If the interval angle is an interval rotation angle, $\Omega = [\underline{\omega}, \overline{\omega}]$, its width can be any nonnegative value. If the interval angle is an interval directed angle, $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, its width, $w(\Theta)$, will be less than 2π. While it is possible to use an interval with negative width to represent an exterior angle, this adds unnecessary complexity without any corresponding benefit. To uniquely define the relationship between an interval rotation angle and its corresponding interval directed angle requires that two rotation counters, k and l, be used, one for the infimum and one for the supremum. Therefore the relationship between an interval rotation angle and its corresponding interval directed angle is:

$$\Omega = \left[\underline{\theta} + k2\pi, \overline{\theta} + l2\pi\right], \tag{9}$$

for particular integers, k and l. The choice of k and l, depends on the convention used to relate rotation and direction angles. In addition, k and l must be chosen so

$$0 \le w(\Theta) < 2\pi. \tag{10}$$

This inequality can be used to impose a constraint on $l-k$. From (9) and the definition of the width of an interval,

$$w(\Omega) = w(\Theta) + (l-k)\,2\pi. \tag{11}$$

Solving for $w(\Theta)$ and substituting in (10), yields:

$$0 \le w(\Omega) - (l-k)\,2\pi < 2\pi, \text{ or} \tag{12a}$$

$$0 \le \frac{w(\Omega)}{2\pi} - (l-k) < 1. \tag{12b}$$

But

$$0 \le x - \lfloor x \rfloor < 1. \tag{13}$$

Therefore,

$$l-k = \left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor. \tag{14}$$

The four different conventions used to relate rotation and direction angles can each be applied either to $\underline{\omega}$, $\overline{\omega}$, or to the midpoint of Ω, $m(\Omega)$. This will result in a second function of k or l or both, from which both k and l can be uniquely determined. For example, from the definition of Θ in (9) and the definition of the midpoint of an interval:

$$m(\Omega) = m(\Theta) + (k+l)\,\pi. \tag{15}$$

Applying (1) to $m(\Omega)$ preserves the sign of $m(\Omega)$ in $m(\Theta)$:

$$m(\Theta) = m(\Omega) - \operatorname{trunc}\left(\frac{m(\Omega)}{2\pi}\right) 2\pi. \tag{16}$$

Solving for $m(\Omega)$, substituting into (15), and solving for $k+l$, yields

$$k+l = 2\operatorname{trunc}\left(\frac{m(\Omega)}{2\pi}\right), \tag{17}$$

from which, with the help of (14), it is trivial to obtain explicit expressions for k and l:

$$k = \operatorname{trunc}\left(\frac{m(\Omega)}{2\pi}\right) - \frac{1}{2}\left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor \tag{18a}$$

$$l = \operatorname{trunc}\left(\frac{m(\Omega)}{2\pi}\right) + \frac{1}{2}\left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor \tag{18b}$$

Another possibility is to apply (4a) to the infimum of Ω:

$$\underline{\theta} = \underline{\omega} \bmod 2\pi = \underline{\omega} - \left\lfloor \frac{\underline{\omega}}{2\pi} \right\rfloor 2\pi, \tag{19}$$

in which case

$$k = \left\lfloor \frac{\underline{\omega}}{2\pi} \right\rfloor. \tag{20}$$

Substituting k into (14) and solving for l, yields:

$$l = k + \left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor. \tag{21}$$

As a final example, substitute $m(\Theta)$ and $m(\Omega)$ for θ and ω in (6a) and use (15) to eliminate $m(\Theta)$. The result is:

$$k + l = 2 \left\lfloor \frac{m(\Omega) + \pi}{2\pi} \right\rfloor, \tag{22}$$

from which, with the help of (14), explicit expressions for k and l follow at once:

$$k = \left\lfloor \frac{m(\Omega) + \pi}{2\pi} \right\rfloor - \frac{1}{2} \left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor \tag{23a}$$

$$l = \left\lfloor \frac{m(\Omega) + \pi}{2\pi} \right\rfloor + \frac{1}{2} \left\lfloor \frac{w(\Omega)}{2\pi} \right\rfloor. \tag{23b}$$

If the above or similar conventions are not used, some interval directed angles cannot be sharply represented. For example, if the restriction is imposed that both the infimum and supremum of every interval directed angle must be confined to the interval $[-\pi, \pi]$, then the only possible interval that contains the angle $\left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]$ is $[-\pi, \pi]$. This result is *not* sharp.

3.1 Trigonometric function conventions

When performing the reduction of large interval rotation angles to compute trigonometric functions, both interval endpoints must be considered or a reduced interval angle with negative width may be produced. For example, if the endpoints of the interval angle $\left[\frac{7}{4}\pi, \frac{9}{4}\pi\right]$ are independently reduced modulo 2π, the result is $\left[\frac{7}{4}\pi, \frac{1}{4}\pi\right]$, which has negative width and is therefore inconsistent with the original rotation angle. Applying any of the conventions described above precludes the possibility of reduced interval angles with negative width.

Interval enclosures of the trigonometric functions and their inverses are defined using the same point conventions established for the principle radian measure intervals.

3.2 Interval Enclosure of Fortran `ATAN2(Y,X)` function

For a sharp interval enclosure of the Fortran `ATAN2(Y,X)` function denoted by Θ, a convention must be chosen with which to uniquely define all possible returned interval directed angles. A number of possible choices exist to resolve what to do when $\overline{x} < 0$ and $0 \in Y$, including:

$$-\pi < \inf(\Theta) \leq \pi, \tag{24a}$$

$$0 \leq \inf(\Theta) < 2\pi, \text{ or} \tag{24b}$$

$$-\pi < m(\Theta) \leq \pi. \tag{24c}$$

This choice, together with (10) results in a unique definition, of the ATAN2 containment set, $\operatorname{cset}(\texttt{ATAN2}, \{(y_0, x_0)\})$, that the interval directed angle, Θ, must include:

$$\operatorname{cset}(\texttt{ATAN2}, \{(y_0, x_0)\}) = \{\theta \mid y = h \sin\theta \in Y,\ x = h\cos\theta \in X\}. \tag{25}$$

See [2] for the definition of a containment set.

Any convention that does not violate (10) is valid. A convention that leads to sharp interval enclosures of any directed angle is acceptable. A convention that is symmetric about zero is consistent with the spirit of the ATAN2 function definition. These three considerations lead to the choice of the constraint in (24c), which is only biased toward positive angles when $m(\Theta) = \pi$. The following table contains the tests and the arguments of the standard ATAN2 function that can be used to compute the endpoints of Θ in an algorithm that satisfies the three chosen constraints. The first two columns define the cases to be distinguished. The third column contains the range of possible values of $m(\Theta)$. The last two columns show how the endpoints of Θ are computed, using the standard ATAN2 intrinsic function. Of course, directed rounding will need to be employed to guarantee containment.

Y	X	$m(\Theta)$	$\underline{\theta}$	$\overline{\theta}$
$-\underline{y} < \overline{y}$	$\overline{x} < 0$	$\frac{\pi}{2} < m(\Theta) < \pi$	$\texttt{ATAN2}(\overline{y}, \overline{x})$	$\texttt{ATAN2}(\underline{y}, \overline{x}) + 2\pi$
$-\underline{y} = \overline{y}$	$\overline{x} < 0$	$m(\Theta) = \pi$	$\texttt{ATAN2}(\overline{y}, \overline{x})$	$2\pi - \underline{\theta}$
$\overline{y} < -\underline{y}$	$\overline{x} < 0$	$-\pi < m(\Theta) < -\frac{\pi}{2}$	$\texttt{ATAN2}(\overline{y}, \overline{x}) - 2\pi$	$\texttt{ATAN2}(\underline{y}, \overline{x})$

The following examples illustrate the difference between the three conventions in (24).

Example 1 *For* $Y_1 = [-1, 0]$ *and* $X = [-1, -1]$:

$$\Theta_1 = \begin{cases} \left[\pi, \frac{5}{4}\pi\right] & \textit{using (24a),} \\ \left[\pi, \frac{5}{4}\pi\right] & \textit{using (24b), and} \\ \left[-\pi, \frac{-3}{4}\pi\right] & \textit{using (24c)} \end{cases} \tag{26}$$

Example 2 *For* $Y_2 = [-1, 1]$ *and* $X = [-1, -1]$:

$$\Theta_2 = \begin{cases} \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right] & \textit{using (24a),} \\ \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right] & \textit{using (24b), and} \\ \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right] & \textit{using (24c)} \end{cases} \tag{27}$$

Example 3 *For* $Y_3 = [0, 1]$ *and* $X = [-1, -1]$:

$$\Theta_3 = \begin{cases} \left[\frac{3}{4}\pi, \pi\right] & \textit{using (24a),} \\ \left[\frac{3}{4}\pi, \pi\right] & \textit{using (24b), and} \\ \left[\frac{3}{4}\pi, \pi\right] & \textit{using (24c)} \end{cases} \tag{28}$$

Just as with large rotation angle reduction, imposing the restriction that the interval directed angle $\Theta \subseteq [-\pi, \pi]$ unnecessarily precludes returning sharp intervals that contain π.

When using any convention, inherent ambiguity must be considered when directed angles are compared. For example, if (24c) is used to compare Θ_1 and Θ_2, it can be mistakenly concluded that $\Theta_1 \cap \Theta_2 = \emptyset$, the empty interval. Just as in the case of points, the appropriate number of rotations must be added or subtracted from one or the other interval to preclude this mistake. Either adding 2π to Θ_2 or subtracting 2π from Θ_1 is required in this case.

4 Conclusion

The treatment of both point and interval angles requires care because of the inherent ambiguity that exists in directed angles. To sharply represent all possible directed angles, unnecessary restrictions must be avoided.

References

[1] R. C. James, G. and James. *James and James Mathematics Dictionary.* Van Norstrand Reinhold, 11 Fifth Avenue, New York, New York 10003, fourth edition, 1976.

[2] G. W. Walster. The "Simple" Closed Interval System. Technical report, Sun Microsystems, February 2000.