Dear all,

I was out of town for ten days without access to the internet. Coming home I find plenty of mails on P1788 and it seems to me that a conflict has arisen between two different extensions of conventional interval arithmetic. Since both extensions originated in Karlsruhe I think I should comment a little on them. One extension is aiming for more mathematical regularity (**MR).** The other one for algebraical closedness (**AC).** These are my comments: Let $\mathbb{IR}$ denote the set of non empty, closed and bounded real intervals (conventional interval arithmetic).

MR: The following are well established mathematical results and concepts:
A non empty set M with an associative operation $* : M \times M \to M$

$$(a * b) * c = a * (b * c)$$

is called a semigroup. A semigroup is called regular if the cancellation law holds

$$a * x = b * x \Rightarrow a = b.$$

Theorem: A commutative, regular semigroup M can be embedded into a group G with $M \subseteq G$. For each $b \in M$, G contains its inverse b^{-1}.

Applications:
I. The natural numbers $\{\mathbb{N}, +\}$ are a commutative, regular semigroup. Embedding it into a group leads to the whole numbers $\{\mathbb{Z}, +\}$. Every element $a \in \mathbb{N}$ has an inverse $-a \in \mathbb{Z}$.

II. The set $\{\mathbb{IR}, +\}$ is a commuatative, regular semigroup. For intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \in \mathbb{IR}$ the cancellation law holds: $\boldsymbol{a}+\boldsymbol{x} = \boldsymbol{b}+\boldsymbol{x} \Rightarrow \boldsymbol{a} = \boldsymbol{b}$. So $\{\mathbb{IR}, +\}$ can be embedded into a group $\{\mathbb{GIR}, +\}$ such that each element $\boldsymbol{a} \in \mathbb{IR}$ has an inverse $\boldsymbol{a}^{-1}$ in $\{\mathbb{GIR}, +\}$ with the property $\boldsymbol{a} + \boldsymbol{a}^{-1} = [0, 0]$. For an element $\boldsymbol{a} = [a1, a2]$ the inverse is $\boldsymbol{a}^{-1} = [-a1, -a2]$.

In his Dr. theses (Karlsruhe 1973) Edgar Kaucher shows that and how other operations and concepts like subtraction, multiplication, division, subset, infimum, supremum, continuity can be extended to the new elemets of $\mathbb{GIR}$.

AC: In the real number field division by zero is not defined. The extension **MR** silently assumes that in $\mathbb{IR}$ division by an interval that contains zero is excluded. The extension **AC** in contrary allows division by intervals that contain zero. This leads to closed, but unbounded intervals. Explicit formulas for operations for unbounded intervals can be obtained from those for bounded intervals by continuity considerations. This leads to an algebraically closed calculus that is free of exceptions with the only irregularity that division by an interval that contains zero as an interior point leads to two separate closed but unbounded intervals. But this case also can easily be handled by computers. For details see my book: *Computer Arithmetic and Validity*, de Gruyter 2008, or my article in the proceedings of the Dagstuhl seminar of January 2008.

The conflict that occurs in recent mails seems to be unavoidable. While **AC** allows unbounded intervals, **MR** has to avoid them. For unbounded intervals the cancellation law does not hold which is necessary for application of the above theorem, for instance:

$$[a1, a2] + (-\infty, +\infty) = [b1, b2] + (-\infty, +\infty) \Rightarrow a1 - \infty = b1 - \infty, a2 + \infty = b2 + \infty$$
$$\not\Rightarrow a1 = b1 \text{ and } a2 = b2.$$

Comments on recent mails:
AC has been accepted by P1788. It allows useful applications. Newton's method is one of them. **MR** may allow other useful applications. Matching **AC** and **MR** seems to be impossible.

Best regards
Ulrich