Creating a P1788 Standard for Modal Intervals

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Abstract

Dear Svetoslav, in this paper I sketch my thoughts and ideas about how to create a P1788 standard for modal intervals.

1 Motivation

For P1788 interval arithmetic we have $\overline{\mathbf{I}\mathbb{R}}$ as the set of all closed proper intervals plus the empty set. In John's Level 1 draft text he gives in section 5.4 the following definitions.

For any real function $f(x) : \mathbb{R}^n \to \mathbb{R}$ with natural domain $D_f \subseteq \mathbb{R}^n$, and for any $X \subseteq \overline{\mathbf{I}\mathbb{R}}^n$

$$Range(f, X) \equiv \left\{ \; f(x) : x \in X \wedge x \in D_f \; \right\}. \tag{1}$$

Then John defines the "natural interval extension" of f as

$$f(X) \equiv Hull(Range(f, X)). \tag{2}$$

Remark 1 *Note that (2) is the exact interval range enclosure of $f(x)$ over the interval box X as defined by (1), which ignores the values of x outside the natural domain of f.*

For some digitally-rounded computational program $F(X) \in \overline{\mathbf{I}\mathbb{R}}$ that John calls the "interval extension" of f on X, we then have the well-known "Fundamental Theorem of Interval Arithmetic" (a.k.a. FTIA) that everyone in P1788 is so familiar with:

$$f(X) \subseteq F(X). \tag{3}$$

In my opinion, all of the above is so well-understood and firmly entrenched it is certain no standard will be produced that does not obey or include these definitions. In writing a standard for Kaucher arithmetic, our task (as I see it) is to write a set of definitions that are extensions of those provided by John. This means we must provide equivalent definitions for (1)–(3) that extend the interval system to $\overline{\mathbb{IR}}$, the set of all closed proper and improper intervals plus the empty set.

What follows is a sketch of my ideas of how it will be possible to do this.

2 Exact Form

To me, a very important observation is that definitions (1)–(2) deal exclusively with exact forms. In other words, there is not any identification in these definitions between the inclusion relation of a digitally-rounded or overestimated result and the mathematically exact value. That is entirely the subject of definition (3). Understanding the exact form is necessary even for classical interval arithmetic because some mathematical properties that are valid in exact arithmetic are not valid in a digitally-rounded finite arithmetic. This is also true for Kaucher arithmetic. Therefore, by starting with the exact forms, we define what is mathematically true under the circumstances of no digital rounding errors or other kinds of overestimation.

To ease the progression of the task before us, let us first begin by assuming that the real function f is continuous on the input box X and that we are always dealing with an X that is non-empty and bounded. We will consider what happens when these assumptions are dropped later in this paper. Under this assumption a starting point that is familiar to me is the definition of f^* by Gardenes, et. al.,

$$f^*(X) \equiv \left[\min_{x_p \in X_p} \max_{x_i \in X_i} f(x_p, x_i), \max_{x_p \in X_p} \min_{x_i \in X_i} f(x_p, x_i) \right], \tag{4}$$

where X_p and X_i are the proper and improper sub-components of $X \in \overline{\mathbb{IR}}^n$, respectively. To avoid tedious notation, I also write $x \in X$ as a shortcut for "x is an element of the set represented by the proper or improper interval X."

At this point, we should observe a few critical facts:

- For the arithmetic operations f^* gives the same results as Edgar Kaucher's famous inf-sup formulas. In your SCAN 2006 paper you also presented equivalent mid-rad formulas for Kaucher's formulas. So we know that both the inf-sup and mid-rad presentations of the quasivector-space structures and Kaucher arithmetic are embedded into this generalized formulation provided by f^*.

- In the special case that all components of X are proper intervals, the definition of f^* reduces to

  $$f^*(X) \equiv \left[\min_{x \in X} f(x), \max_{x \in X} f(x) \right]. \tag{5}$$

  This coincides exactly with John's "natural interval extension." In other words both (2) and (5) are different notations for the definition of the same result.

- John's very generalized definition (2) is notably relevant for the extension of any f onto $\overline{\mathbf{I}\mathbb{R}}^n$, not just the arithmetic operations. Similarly, while quasivector-space and Kaucher's formulas are relevant to the arithmetic operations, f^* gives us the most generalized definition for the extension of any f onto $\overline{\mathbb{IR}}^n$.

- Gardenes, et. al. also provide a complementary definition for f^* called f^{**}, but since it is known that for the arithmetic operations and all of the library functions currently to be included in the standard that

$$f^*(X) = f^{**}(X),$$

this makes unnecessary any consideration of including f^{**} within the scope of the standard.

These relevant facts inspire me to suggest that we should focus on f^* as the basis for including Kaucher intervals in the standard. We can present in an appendix both the inf-sup and mid-rad formulations of the arithmetic operations and show how they are special case of f^*. We can also present in the appendix the algebraic structure of the quasivector space, if you wish, and explain how it too is embedded in the definition provided by f^*.

Proposition 2 *As mentioned, a few restrictive assumptions have thus far been made, notably that i) f is continuous on the input box X, and ii) that X is a non-empty and bounded element of $\overline{\mathbb{IR}}^n$. If we can provide suitable definitions that allow us to remove these assumptions when using (4) as the foundation of an interval standard, then we will have successfully provided a complete extension of (1)–(2) for a new interval standard that applies to any conceivably relevant f on $\overline{\mathbb{IR}}^n$. Embedded within this new interval standard are the inf-sup and mid-rad forms of Kaucher's arithmetic and related quasivector spaces.*

I believe that this proposition is true and can be accomplished by a combination of i) necessary mathematical definitions and ii) decorations. I will elaborate on this more in subsequent sections.

3 Semantic Theorem

For classic intervals, John gives (3). We need a similar FTIA for the case of f^*. If we define

$$Q(y, Y) \equiv \begin{cases} (\exists y \in Y) & \text{if } Y \text{ is proper} \\ (\forall y \in Y) & \text{if } Y \text{ is improper} \end{cases} \tag{6}$$

for a modal interval $Y \in \overline{\mathbb{IR}}$, then we have the "Semantic Theorem for f^*" by Gardenes, et. al.:

$$f^*(X) \subseteq F(X) \Leftrightarrow (\forall x_p \in X_p) Q(y, F(X)) (\exists x_i \in X_i) : y = f(x_p, x_i) \tag{7}$$

This is the "Fundamental Theorem of Interval Arithmetic" for Kaucher intervals that we need, because it gives us an identification between the inclusion relation of an overestimated result $F(X)$ of some digitally-rounded computational program and the mathematically exact form $f^*(X)$. Without such a theorem, we cannot know how to correctly implement Kaucher arithmetic in a computer based on a finite approximation of the reals.

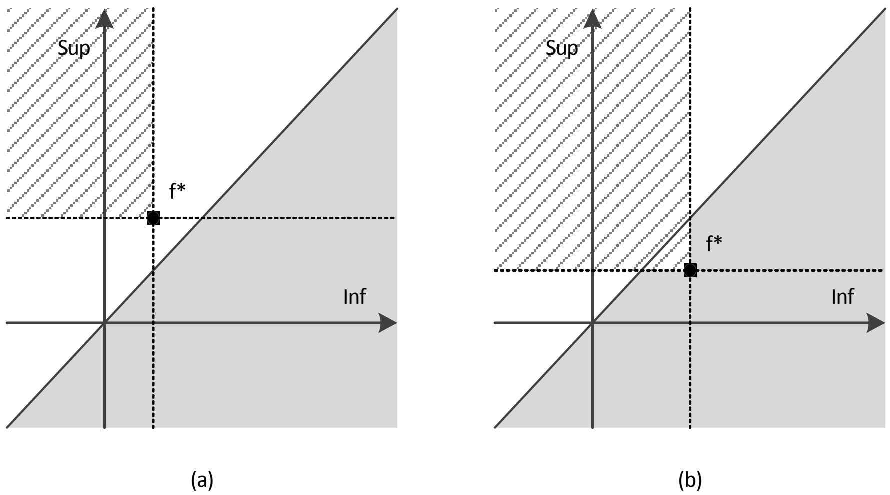

Figure 1: Graphical depiction of the modal interval values of Y in (10) and (11) which cause the Semantic Theorem for f^* to be true. All of the points in the plane above the Inf = Sup line are proper intervals and points below this line in the shaded region are improper intervals. The point f^* is the exact result and any point in the hatched region is a modal interval value of Y that will satisfy the theorem. Solutions for (10) are depicted on the left in (a) and solutions for (11) are depicted on the right in (b). Note that if f^* is a proper interval then Y will always be a proper interval. If f^* is an improper interval then Y may be a proper or improper interval.

Remark 3 *As noted by Vladik K. in the P1788 forum, the universal and existential quantifiers are generally not commutative. However this is precisely the reason (7) takes care to always order the universal quantifiers before the existential quantifiers.*

Example 4 *Lets consider* $sqrt(x)$ *on non-negative* x*. For the proper and improper intervals* $[1, 4]$ *and* $[4, 1]$ *we have*

$$f^*([1,4]) = [1,2] \tag{8}$$
$$f^*([4,1]) = [2,1]. \tag{9}$$

Semantic Theorem for f^* *then introduces quantifiers to define the inclusion relation between these exact results and some computed result* Y*, i. e.,*

$$f^*([1,4]) \subseteq Y \quad \Leftrightarrow \quad (\forall x \in X)Q(y,Y) : y = sqrt(x) \tag{10}$$
$$f^*([4,1]) \subseteq Y \quad \Leftrightarrow \quad Q(y,Y)(\exists x \in X) : y = sqrt(x) \tag{11}$$

where again the notation $Q(y, Y)$ *means* $(\exists y \in Y)$ *if* Y *is proper and* $(\forall y \in Y)$ *if* Y *is improper. Now, in each case (10) and (11), to visualize the values of*

Y which cause the Semantic Theorem to be true, you can plot f^* as a point on the inf-sup plane. This point divides the plane into four quadrants as depicted in Figure 1. Any point in the upper-left quadrant is a modal interval value of Y that will satisfy the theorem. Since (8) is a proper interval, Y in (10) will always be a proper interval so we could simply expand the notation $Q(y, Y)$ and write

$$f^*([1,4]) \subseteq Y \Leftrightarrow (\forall x \in X)(\exists y \in Y) : y = sqrt(x). \tag{12}$$

On the other hand, since (9) is an improper interval, Y in (11) may be a proper or improper interval, hence we can expand the notation $Q(y, Y)$ only after knowing the value of Y. For example, if $Y = [1.9, 1.2]$ then we have

$$f^*([4,1]) \subseteq Y \Leftrightarrow (\forall y \in Y)(\exists x \in X) : y = sqrt(x),$$

but if $Y = [-1, 4]$ then we have

$$f^*([4,1]) \subseteq Y \Leftrightarrow (\exists y \in Y)(\exists x \in X) : y = sqrt(x).$$

In all cases, the predictions made by the Semantic Theorem are true.

As the example illustrates, Semantic Theorem for f^* tells us how we may correctly round the exact results. For example, a computer program could implement the *sqrt* operation in finite floating-point arithmetic with the formula

$$sqrt([a, b]) \equiv [roundDown(sqrt(a)), roundUp(sqrt(b))]. \tag{13}$$

Because of the Semantic Theorem, we know this formula is mathematically correct for any proper or improper interval input.

Remark 5 *The critical fact to notice is that John's definition (3) is simply the special-case of (7) when all the components of X are proper and X is in the natural domain of f. Even the digitally-rounded formula (13) given to us by the Semantic Theorem for f^* is the same computer implementation required by (3)! The only difference is that John's definition (3) arbitrarily prohibits the input to be an improper interval, even though (13) is valid for both proper and improper interval inputs.*

4 Issues to be Addressed

The previous sketch shows that the significant issues to be addressed are:

- Following John's definition (2) of the "natural interval extension" of a real function, how can the definition (4) of f^* be similarly defined to ignore values outside the natural domain of f?

- How must Semantic Theorem for f^* be extended to properly handle empty and unbounded intervals?

My belief is that unbounded intervals are not going to be a problem. In Motion 12, the P1788 committee already accepted the idea that undefined values such as $\infty - \infty$ which may arise in the modal arithmetic can be handled with decorations. I think it will be wise to follow in this direction.

Empty intervals pose a unique problem to the Semantic Theorem since any existentially-quantified proposition on the empty set is always false. I believe in this case we should be careful to follow P1788's progress on decorations and provide definitions that will be consistent.

Finally I would note that for users who wish to perform "pure" Kaucher arithmetic on intervals that are non-empty and bounded, decorations provide a mechanism to accomplish this. For example, decorations can prove that a lengthy computation performed in a computer involved only operations that were performed on a continuous domain. This is another point that can be elaborated on.