

MOTION 13, VERSION 4

COMPARISON RELATIONS FOR AN INTERVAL ARITHMETIC STANDARD

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ABSTRACT. This proposal deals with the definition of comparison relations for an interval arithmetic computation standard and it considers their most important properties. Simplicity is a general requirement for this standard. So the number of comparison relations defined by the standard should be kept to a minimum. This proposal considers seven elementary comparison relations for intervals over the floating-point numbers.

KEY WORDS: computer arithmetic, floating-point arithmetic, interval arithmetic, arithmetic standards.

1. INTRODUCTION

Diverse programming environments for interval arithmetic have been developed during the last four decades. Several of these define a large number of comparison relations. This raises the question which of these relations should be required and specified by an interval arithmetic standard. Simplicity is a general requirement for this standard. So the number of comparison relations defined by the standard should be kept to a minimum. This is not a severe restriction. Comparison relations for intervals are easy to define and easy to execute. So if a user needs a particular relation for his application an ad hoc definition will suffice. The saying *a good theory is the best practice* may help making the right selection. From a theoretical point of view three comparison relations for intervals are fundamental for applications of interval arithmetic. They are: *equality*, *subset of*, and *less than or equal*. So this proposal takes particular care of these three relations. Their most important properties are considered. Explicit formulas for interval operations over the real and complex numbers and for intervals of vectors and matrices over the real and complex numbers can be developed by using these three comparison relations. See [5].

To a certain extent selecting the basic comparisons is a matter of taste. To keep the standard simple the original proposal required only 4 relations as elementary comparisons. Many others can be put together using these 4 relations.

Several very interesting comments were made by a number of colleagues. One by Nate Hayes [2] convinced me to revise the motion. In a mail of April 11 he suggests to extend the 4 comparisons in the original proposal by 3 additional relations. Indeed many derived comparisons can be expressed simpler by the 7 relations than by the original 4. In particular Nate Hayes shows in Table 2 of [2] that all certainly and possibly relations of Sun's interval Fortran can be expressed by very simple expressions with the 7 suggested relations.

This revised Motion 13, Version 4 now contains 7 comparisons as elementary relations for intervals. Implementations of the standard shall support the seven comparison relations defined in Section 2 below.

In the following \mathbb{R} denotes the set of real numbers and $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$. With the order relation \leq , $\{\mathbb{R}^*, \leq\}$ is a complete lattice, i.e., every subset has an infimum and a supremum. \mathbb{F} denotes the set of finite floating-point numbers of a given format and encoding and $\mathbb{F}^* := \mathbb{F} \cup \{-\infty, +\infty\}$. $\overline{\mathbb{R}}$ denotes the set of all intervals as defined by motion 3, that is

all closed connected subsets of \mathbb{R} including \emptyset . \mathbb{IR} denotes the nonempty, bounded members of \mathbb{IR} . \mathbb{IF} denotes the subset of \mathbb{IR} with bounds of \mathbb{F} . \mathbb{IF} denotes the subset of \mathbb{IR} comprising \emptyset together with all intervals whose bounds are in \mathbb{F}^* .

2. COMPARISON RELATIONS AND LATTICE OPERATIONS

If \mathbf{a} and \mathbf{b} are intervals of \mathbb{IR} with bounds $a_1 \leq a_2$ and $b_1 \leq b_2$ respectively, then these 7 relations are defined by:

| <i>Symbol</i> | <i>Definition</i> | <i>Implementation</i> (for nonempty \mathbf{a}, \mathbf{b}) |
|-----------------------------------|--|--|
| $\mathbf{a} = \mathbf{b}$ | \mathbf{a} equals \mathbf{b} as sets | $a_1 = b_1 \wedge a_2 = b_2$ |
| $\mathbf{a} \subseteq \mathbf{b}$ | \mathbf{a} is a subset of \mathbf{b} | $b_1 \leq a_1 \wedge a_2 \leq b_2$ |
| $\mathbf{a} \leq \mathbf{b}$ | \mathbf{a} is less than or equal to \mathbf{b} | $a_1 \leq b_1 \wedge a_2 \leq b_2$ |
| $\mathbf{a} \preceq \mathbf{b}$ | \mathbf{a} precedes or touches \mathbf{b} | $a_2 \leq b_1$ |
| $\mathbf{a} \subset \mathbf{b}$ | \mathbf{a} is interior to \mathbf{b} | $b_1 < a_1 \wedge a_2 < b_2$ |
| $\mathbf{a} < \mathbf{b}$ | \mathbf{a} is less than \mathbf{b} | $a_1 < b_1 \wedge a_2 < b_2$ |
| $\mathbf{a} \prec \mathbf{b}$ | \mathbf{a} precedes \mathbf{b} | $a_2 < b_1$ |

Since bounds for intervals of \mathbb{IR} may be $-\infty$ or $+\infty$ these comparison relations are executed as if performed in the lattice $\{\mathbb{R}^*, \leq\}$.

Equality is an equivalence relation (symmetric, reflexive, transitive), *subset of* and *less than or equal* are partial orders (reflexive, transitive, antisymmetric). These three relations are most fundamental for all interval computations.

If in the relations $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \preceq \mathbf{b}$, $\mathbf{a} < \mathbf{b}$, $\mathbf{a} \prec \mathbf{b}$ any operand is the empty set the result is false.

With the relation \subseteq , $\{\mathbb{IR}, \subseteq\}$ is a lattice. The least element in $\{\mathbb{IR}, \subseteq\}$ is the empty set \emptyset and the greatest element is the set $\mathbb{R} = (-\infty, +\infty)$. The infimum of two elements $\mathbf{a}, \mathbf{b} \in \mathbb{IR}$ is the intersection and the supremum is the interval hull (convex hull):

$$\begin{aligned} \inf(\mathbf{a}, \mathbf{b}) &= \mathbf{a} \cap \mathbf{b} := [\max(a_1, b_1), \min(a_2, b_2)] \quad \text{or the empty set } \emptyset, \\ \sup(\mathbf{a}, \mathbf{b}) &= \mathbf{a} \cup \mathbf{b} := [\min(a_1, b_1), \max(a_2, b_2)]. \end{aligned}$$

$\{\mathbb{IF}, \subseteq\}$ is a sublattice of $\{\mathbb{IR}, \subseteq\}$.

With the order relation \leq , $\{\mathbb{IR} \setminus \{\emptyset\}, \leq\}$ is a conditionally complete¹ lattice. In this order, the lattice operations infimum and supremum of two elements $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \setminus \{\emptyset\}$, here called glb (*greatest lower bound*) and lub (*least upper bound*) are the intervals

$$\begin{aligned} \text{glb}(\mathbf{a}, \mathbf{b}) &:= [\min(a_1, b_1), \min(a_2, b_2)], \\ \text{lub}(\mathbf{a}, \mathbf{b}) &:= [\max(a_1, b_1), \max(a_2, b_2)], \end{aligned}$$

respectively.

$\{\mathbb{IF} \setminus \{\emptyset\}, \leq\}$ is also a lattice. The greatest lower bound and the least upper bound of two elements $\mathbf{a}, \mathbf{b} \in \{\mathbb{IF} \setminus \{\emptyset\}, \leq\}$ are the same as in $\{\mathbb{IR} \setminus \{\emptyset\}, \leq\}$. The least element of $\{\mathbb{IF} \setminus \{\emptyset\}, \leq\}$ is the interval $(-\infty, \text{minreal}]$ and the greatest element is $[\text{maxreal}, +\infty)$.

Comment: A particular case of inclusion is the relation *element of*. For $a \in \mathbb{R}$ it is defined by

$$a \in \mathbf{b} :\Leftrightarrow b_1 \leq a \wedge a \leq b_2.$$

¹Every bounded subset has an infimum and a supremum.

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