Decorated Intervals
A Pragmatic Approach

Version 1.2

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1 The Motion (Draft)

1.1 Definitions

Intervals are sets of real numbers (motion 3). As such sets they are comparable with respect to containment.

Decorations are properties that carry information on the history of the interval. They are a measure of confidence of intervals.

A decoration is determined when a function f is applied on an interval (box) $\mathbf{x}$.

P1788 shall provide the following 4 decorations :

Definition 1 .

name	*abbrev*	*definition*
safe	*saf*	$\mathbf{x} \subseteq D_f$ *f continuous on* $\mathbf{x}$
defined	*def*	$\mathbf{x} \subseteq D_f$ *f not continuous on* $\mathbf{x}$
enclosing	*enc*	$\mathbf{x} \cap D_f \neq \emptyset$ *and* $\mathbf{x} \cap (\mathbb{R} \setminus D_f) \neq \emptyset$
nowhere defined	*ndf*	$\mathbf{x} \cap D_f = \emptyset$

Here D_f denotes the domain of f.

Definition 2 *Decorations are ordered with respect to the confidence level:*
$ndf < enc < def < saf$

Definition 3 *A decorated interval is a pair of an interval and a decoration. The set of all decorated intervals is called* $\overline{\mathbb{DIR}}$

Remark 1 *We emphasize, that the decoration does not contain information about the contents of the interval but only tracks properties of the environment in which the interval has been computed. Nevertheless it is reasonable to put an interval together with a decoration.*

If it is appropriate we denote the interval part of a decorated interval a "bare interval" and the decoration part is called a "bare decoration".

In the following we assume that every interval is decorated. Nevertheless applications may totally ignore decorations, since for each operation the resulting bare interval depends only on the bare input intervals, see the next subsection.

1.2 Assertions in $\overline{\mathbb{IR}}$

For bare intervals we have the fundamental assertions:

Definition 4 *Interval Extension*

A total function $\mathbf{F} : \overline{\mathbb{IR}}^k \Rightarrow \overline{\mathbb{IR}}$ is called an interval extension of a (partial) function $f : \mathbb{R}^k \Rightarrow \mathbb{R}$, if for all possible intervals $\mathbf{x}_i$ and values $x_i \in \mathbf{x}_i$ we have $f(x_1, \ldots, x_k) \in \mathbf{F}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$

Proposition 1 *An interval extension $\mathbf{F}(\mathbf{x})$ of an expression f encloses the range over $\mathbf{x}$ of the point function defined by the expression f.*

Theorem 1 *Subset Property or Isotonicity*

Let $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{IR}}$ and an interval extension $\mathbf{F} : \overline{\mathbb{IR}} \Rightarrow \overline{\mathbb{IR}}$ of a continuous function f. Then

$\mathbf{x} \subseteq \mathbf{y} \implies \mathbf{F}(\mathbf{x}) \subseteq \mathbf{F}(\mathbf{y})$

Theorem 2 *Enclosure Property or FTIA*

The particular interval function $\mathbf{F}$ that results from applying straight-forward interval computation onto the expression f is an interval extension.

If every variable x_i occurs only once in an expression for a continuous function, and if all subexpressions compute their range, then $\mathbf{F}$ computes the exact range.

We are going to transfer these theorems to the space of decorated intervals $\overline{\mathbb{DIR}}$.

1.3 Arithmetic in $\overline{\mathbb{DIR}}$

Remark 2 *The arithmetic of bare intervals is defined in motion 5 and 10.2 (now replaced by Table 5.6.2 of draft 03.2).*

In particular applying an arithmetic function to an empty argument produces an empty set: $f(\emptyset) = \emptyset$

Definition 5 *The interval parts are computed like bare intervals, the results do not depend on any decorations.*

1.3.1 Generation and Propagation of Decorations

We model the computation of a decorated interval expression with a decorated expression tree (DET).

Definition 6 *Initialization*
A literal interval constant is given the decoration saf. A constructor builds a decorated interval with default decoration saf.

Definition 7 *Decorated Expression Tree*
*A decorated interval is a **decorated expression tree**. If $(t_i, d_i), i = 1, \ldots k$ are DETs and f is a k-ary decorated function then $t = ((f, d_f), (t_1, d_1), \ldots (t_k, d_k))$ is a DET where d_f is the decoration that is generated by the call $f(t_1, \ldots t_k)$ according to table 1.*

Definition 8 *Propagation Rule*
For the decorated expression tree $t = ((f, d_f), (t_1, d_1), \ldots (t_k, d_k))$ we use the following evaluation rule:

$$(\mathbf{y}, d_y) = (f(t_1, \ldots t_k), \min\{d_f, d_1, \ldots d_k\}) \tag{1}$$

Usually, we trust our input intervals, but then we always track the worst possible decoration.

Theorem 3 *Obviously we will compute a lower bound of the decoration, if we only apply rule (1).*

There is no chance to overlook a case where we incorrectly promise an unstained path.

1.3.2 Determining the Decoration

Continuous functions

The generated decoration is always saf even in the case of over- or underflow. Hence, the minimal decoration of the inputs is propagated.

Functions with isolated singularities

If $\mathbf{x}$ does not contain a singularity, set $d_f = saf$, else if $\mathbf{x}$ is a point interval produce ndf, in this case the bare interval result has to be $\emptyset$. Otherwise return enc.

$$\begin{aligned} 1/[0,0] &= (\emptyset, ndf) = 1/\emptyset \\ 1/[0,1] &= ([1,\infty], enc) \\ 1/[\varepsilon,1] &= ([1, 1/\varepsilon], saf) \\ 1/[\varepsilon^2,1] &= ([1,\infty], saf) \end{aligned}$$

Non-continuous functions

Generate *def* or *saf* according to the overlapping situation of $\mathbf{x}$ and discontinuity points.

$$\begin{aligned} floor([1,9],saf) &= ([1,9],def) \\ floor([1,9.5],saf) &= ([1,9],def) \\ floor([1,1],saf) &= ([1,1],def) \\ floor([1.25,1.5],saf) &= ([1,1],saf) \end{aligned}$$

Partial functions

Generate decorations according to the overlapping situation of $\mathbf{x}$ and D_f.

$$\begin{aligned} sqrt(\emptyset) = sqrt([-4,-1]) = sqrt([-4,-1],saf) &= (\emptyset, ndf) \\ sqrt([-4,+1]) = sqrt([0,1],enc) &= ([0,1],enc) \\ sqrt([0,1],saf) &= ([0,1],saf) \\ sqrt(floor([0,1])) = sqrt([0,1],def) &= ([0,1],def) \end{aligned}$$

which implies

$$\begin{aligned} &\emptyset \subseteq [-4,-1] \subseteq [-4,1] \implies \emptyset \subseteq (\emptyset, ndf) \subseteq ([0,1],enc) \\ &\emptyset \subseteq [0,1] \subseteq [-4,1] \implies \emptyset \subseteq ([0,1],saf) \subseteq ([0,1],enc) \\ &\emptyset \subseteq ([0,1],def) \subseteq ([0,1.5],def) \\ &\emptyset \subseteq ([0,1],def) \subseteq ([0,1],def) \end{aligned}$$

1.3.3 Treatment of $\emptyset$

Remark 3 *The empty set may get all decorations. The intersection of safe, disjoint intervals is a safe empty set e.g. if that set is input to an additon a nowhere defined empty set is generated.*

Definition 9 *Propagation of* $\emptyset$
All k-ary arithmetic functions deliver the result $f(\emptyset) = (\emptyset, ndf)$, *if one component of the inbut box is empty.*

Proposition 2 *The same has to hold for minimum/maximum.*

Proof: We show by a counter example given by Vincent Lefevre that $min(\mathbf{x}, \emptyset) = \mathbf{x}$ leads to a contradiction.

Let $\mathbf{a} = [-2, -1] \subset \mathbf{b} = [-2, 0]$ but for $f(\mathbf{x}) = min(sqrt(\mathbf{x}), [3, 4])$ we have that $f(\mathbf{a}) = [3, 4] \not\subset f(\mathbf{b}) = [0, 0]$.

But if we use a min that returns the empty set we have:

$f(\mathbf{a}) = \emptyset \subset f(\mathbf{b}) = [0, 0]$.

□

1.3.4 Additional functions

Additionally intersection, union, interval hull and max, min can be used as lattice operations defined in motion 13.4.

Intersection with an empty set delivers the empty set, as usual, the decoration is *ndf*. Intersection of two non-empty intervals also propagates the minimum of the input decorations. Note that we have $(\emptyset, saf)$ for disjoint intervals.

Union and hull are not propagating the empty set. Since union is defined everywhere, it does not alter the decorations of its input arguments whose minimum is appropriately chosen.

The interval hull of 2 numbers or intervals, however, is considered to construct a new safe interval, i.e. the decoration is always *saf*.

Note that in contrast to the 754 standard, we require $max(\mathbf{x}, \emptyset) = \emptyset$.

Piecewise continuous functions like *floor* are also allowed.

User defined functions may be called, if they belong to one of the 4 categories introduced in the preceding subsection. There is no problem, if the new function uses welll known functions.

1.3.5 Comparisons

Intervals can be compared with the overlapping relation or the comparisons provided by motion 13.4

Decorated intervals are compared by their bare intervals only, the decorations have no influence. They give information on the history of the executon path, not on the interval itself.

Hence, we can easily repeat the 2 fundamental theorems for decorated intervals.

1.3.6 Bare decorations

Definition 10 *Bare decorations d are promoted to the interval* $([-\infty\,, +\infty], d)$

Bare decorations have lost all information of the interval. In this form all the rules defined above are valide.

1.4 Asssertions in $\overline{\mathbb{DIR}}$

Definition 11 *Interval Extension*

A total function $\mathbf{F} : \overline{\mathbb{DIR}}^k \Rightarrow \overline{\mathbb{DIR}}$ *is called a* ***decorated*** *interval extension of a (partial) function* $f : \mathbb{R}^k \Rightarrow \mathbb{R}$*, if for all possible intervals* $\mathbf{x}_i$ *and values* $x_i \in \mathbf{x}_i$ *we have* $f(x_1, \ldots, x_k) \in \mathbf{F}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$

Proposition 3 *The interval part of a decorated interval extension* $\mathbf{F}(\mathbf{x})$ *of an expression* f *encloses the range over* $\mathbf{x}$ *of the point function defined by the expression* f*.*

Theorem 4 *Subset Property or Isotonicity for* $\overline{\mathbb{DIR}}$

Let $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{DIR}}$ *and a decorated interval extension* $\mathbf{F} : \overline{\mathbb{DIR}} \Rightarrow \overline{\mathbb{DIR}}$ *of a continuous function* f*. Then*

$\mathbf{x} \subseteq \mathbf{y} \implies \mathbf{F}(\mathbf{x}) \subseteq \mathbf{F}(\mathbf{y})$

Theorem 5 *Enclosure Property for Decorated Intervals or FTDIA*

The particular decorated interval function $\mathbf{F}$ *that results from applying straightforward interval computation onto the expression* f *is a decorated interval extension.*

If every variable x_i *occurs only once in an expression for a continuous function, and if all subexpressions compute their range, then* $\mathbf{F}$ *computes the exact range.*

Since the interval part does not depend on decorations, and since the comparisons also do not depend on decorations the theorems are the 2 well-known theorems for bare intervals.

2 Rationale

In the beginning of the motion, we described decorated intervals as relatively independent pieces of information. That is similar to the global discontinuity bit used in filib++. Of course we do not want to stick with global flags, we are clearly in favor of the decoration system described in motions 8 and 15 and in the currently discussed motions 25-A1 and 26. This position paper is our contribution to that discussion.

Our main concern is that the decoration systems of motion 25 and , even more that of 26 are to sophisticated and complicated to be broadly applied or implemented.

Therefore we propose a pragmatic approach, relatively simple, and with understandable implementation rules.

2.1 Decorations

Intervals are sets of real numbers (motion 3) decorated with 1 of the 4 to 7 values. Formally a decoration is a pair of a function f and an interval $\mathbf{x}$ for which the listed property is valid.

Alternative views on decorations

1. decorations are classes of functions (or function spaces) whose members may be called when evaluating an expression.
2. decorations are predicates related to a function and an interval.
3. decorations are properties that carry information on the history of the interval.
4. decorations are properties of the called function.
5. decorations are a measure of confidence of intervals
6. decorations and intervals are equivalent parts of a decorated interval
7. decorations signal exceptions
8. decorations are global flags

The second and 3rd variant are used in motion 26 or 25, respectively. together with 6.

Our approach also favors 3, together with 5.

Decorations are linearly ordered by strength, quality or trustworthiness.

In the two current approaches decorations additionally form a partial order with respect to containment. We don't need that order.

Arithmetic operations and lattice operations have to be extended for decorated intervals.

2.2 The essence of our motion:

Remark 4 *We emphasize again, that the decoration does not contain information about the contents of the interval but only gives some hints on the environment in which the interval has been computed. Nevertheless it is reasonable to put an interval together with a decoration.*

Proposition 4 *Therefore, enclosure of intervals has nothing to do with decorations. We do NOT need a version of the FTIA concerning decorations.*