The following are well established mathematical concepts and results:
A non empty set M with an associative operation $* : M \times M \to M$

$$(a * b) * c = a * (b * c)$$

is called a semigroup. A semigroup is called regular if the cancellation law holds

$$a * x = b * x \Rightarrow a = b.$$

Theorem: A commutative, regular semigroup M can uniquely be embedded into a smallest group G with $M \subseteq G$. For each $b \in M$, G contains its inverse b^{-1}.

Applications:
I. The natural numbers $\{\mathbb{N}, +\}$ are a commutative, regular semigroup. Embedding it into a group leads to the whole numbers $\{\mathbb{Z}, +\}$. Every element $a \in \mathbb{N}$ has an inverse $-a \in \mathbb{Z}$.
II. The set $\{\mathbb{IR}, +\}$ is a commuatative, regular semigroup. For intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \in \mathbb{IR}$ the cancellation law holds: $\boldsymbol{a} + \boldsymbol{x} = \boldsymbol{b} + \boldsymbol{x} \Rightarrow \boldsymbol{a} = \boldsymbol{b}$. So $\{\mathbb{IR}, +\}$ can be embedded into a group $\{\mathbb{I}^*\mathbb{R}, +\}$ such that each element $\boldsymbol{a} \in \mathbb{IR}$ has an unique additive inverse $inv\boldsymbol{a}$ in $\{\mathbb{I}^*\mathbb{R}, +\}$ with the property $\boldsymbol{a} + inv\boldsymbol{a} = [0, 0]$.

For an element $\boldsymbol{a} = [a1, a2] \in \mathbb{I}^*\mathbb{R}$ the inverse is $inv\boldsymbol{a} = [-a1, -a2]$. For each element $a \in \boldsymbol{a}$ its additive inverse $-a$ is an element of $inv\boldsymbol{a}$ and vice versa. The set $\mathbb{I}^*\mathbb{R}$ consissts of the closed and bounded real intervals of $\mathbb{IR}$ and their additive inverses. They are called proper and improper intervals. Thus $\mathbb{I}^*\mathbb{R}$ is just a set of ordered pairs of real numbers

$$\mathbb{I}^*\mathbb{R} := \{[a, b] | a \in \mathbb{R}, b \in \mathbb{R}\}.$$

In his theses (1973) Edgar Kaucher shows that and how operations and concepts like addition, subtraction, multiplication, division, membership, subset, less or equal, union, intersection, continuity, metric, and norm can be extended to the new elements of $\mathbb{I}^*\mathbb{R}$. The result is what in the literature now frequently is called *Kaucher arithmetic*.

Arithmetics in $\mathbb{IR}$ and $\mathbb{I}^*\mathbb{R}$ silently assume that division by an interval that contains zero is not permitted.

The arithmetic of $\mathbb{IR}$, however, can be extended in such a way that division by an interval that contains zero is permitted. This leads to closed, but unbounded real intervals. Explicit formulas for operations for unbounded intervals can be obtained from those for bounded intervals by continuity considerations. This leads to an algebraically closed calculus that is free of exceptions with the only irregularity that division by an interval that contains zero as an interior point leads to two separate closed but unbounded intervals. But this case also can easily be handled by computers. For details see my book: *Computer Arithmetic and Validity*, De Gruyter 2008. The extended set is denoted by $\overline{\mathbb{IR}}$. Arithmetic of $\overline{\mathbb{IR}}$ has been accepted by the IEEE standards committee P1788 for interval arithmetic. It allows useful applications. The extended interval Newton method for computing all zeros of a function in a given interval is one of them. On the computer arithmetic of $\overline{\mathbb{IR}}$ is approximated by arithmetic of $\overline{\mathbb{IF}}$. Since unbounded real intervals are regular elements of $\overline{\mathbb{IR}}$ and $\overline{\mathbb{IF}}$ there is no underflow and no overflow in the calculus of $\overline{\mathbb{IF}}$.

While arithmetic in $\overline{\mathbb{IR}}$ allows unbounded intervals, arithmetic in $\mathbb{I}^*\mathbb{R}$ has to avoid them. For unbounded intervals the cancellation law does not hold which is necessary for the proof and for application of the above theorem, for instance:

$$[a1, a2] + (-\infty, +\infty) = [b1, b2] + (-\infty, +\infty) \Rightarrow a1 - \infty = b1 - \infty, a2 + \infty = b2 + \infty$$
$$\not\Rightarrow a1 = b1 \text{ and } a2 = b2.$$

On the computer, of course, arithmetic of $\mathbb{I}^*\mathbb{R}$ has to be approximated by arithmetic of $\mathbb{I}^*\mathbb{F}$. To avoid underflow and overflow attempts have been undertaken extending arithmetic of $\mathbb{I}^*\mathbb{R}$ by unbounded intervals in a similar manner than arithmetic of $\mathbb{IR}$ to $\overline{\mathbb{IR}}$. While the latter extension can be done with no exceptions the new attempt leads to unreasonable arithmetic operations like $\infty - \infty$ or ∞/∞ which like in IEEE 754 arithmetic only can be set to NaN. Nevertheless, surprisingly the extended interval Newton method has been programmed and successfully executed in the extended set $\overline{\mathbb{I}^*\mathbb{R}}$. The reason for this effect is that the entire calculation only uses arithmetic of $\overline{\mathbb{IR}}$ which is free of exceptions. Since arithmetic of $\mathbb{IR}$ is contained in the arithmetic of $\mathbb{I}^*\mathbb{R}$ the exception free arithmetic of $\overline{\mathbb{IR}}$ naturally is contained in a corresponding arithmetic of $\overline{\mathbb{I}^*\mathbb{R}}$. In its entirety the latter is no longer free of exceptions.

IEEE 754 arithmetic recommends rounding an overflow to $+\infty$ or $-\infty$ and continue the computation. This might end in a useful answer. But this is just speculation. In most cases the result is NaN. All this is not mathematics and it must be kept out of interval arithmetic and of arithmetic in $\mathbb{I}^*\mathbb{R}$. Since both are aiming for guaranteed results they must strictly be kept on mathematical grounds.

It may well be that many applications of arithmetic in $\mathbb{I}^*\mathbb{R}$ also can be solved by using interval arithmetic of $\mathbb{IR}$ or by analysing the expression and computing the lower and the upper bound of an enclosure by case distinctions in floating-point arithmetic. By two reasons, however, this is not a desirable goal:

1. Interval arithmetic can be interpreted as an automatic calculus to deal with inequalities. An algorithm or method described in interval arithmetic in closed form (without the left and right bounds) automatically transforms otherwise necessary case distinctions to the preprogrammed interval operations. The calculus of interval arithmetic thus develops its own dynamics.
2. It can be expected that interval arithmetic will be supported by the computers hardware as soon as the standard IEEE 1788 is generally accepted. Nearly everything that is needed for it is already available on current Intel X86 or AMD processors for multimedia applications. Hardware for interval arithmetic computes the lower and the upper bound of the result of an operation by parallel units simultaneously. With these circuits interval arithmetic is almost as fast as simple floating-point arithmetic. With slightly different control signals for selecting bounds from the operands arithmetic of $\mathbb{I}^*\mathbb{R}$ could be performed with more or less the same circuitry that is available for the arithmetic of $\mathbb{IR}$. This means a great speed advantage for arithmetic of $\mathbb{I}^*\mathbb{R}$ in comparison with an analysis of the expression by hands on methods and a separate computation of the lower and the upper bound in floating-point arithmetic.

The operands of $\mathbb{I}^*\mathbb{R}$ are closed and bounded real intervals of $\mathbb{IR}$ and their additive inverses. On the computer proper and improper floating-point intervals are used instead. The operations for computing the bounds of the result are performed with directed roundings. From an algebraic point of view there is no room for unbounded improper intervals in $\mathbb{I}^*\mathbb{R}$. Now the question remains how to continue computing in $\mathbb{I}^*\mathbb{F}$ in case of an overflow? An answer is: Not at all in case of an improper interval. Mathematically an overflow means that the available range of numbers does not suffice to solve the given problem. The situation can only be cured mathematically by scaling the problem. This is always possible. (On an analog computer during a computation all data must be kept between −1 and +1!). Another solution could be: Use a larger data format for instance.

The mathematical structure of arithmetic in $\mathbb{I}^*\mathbb{R}$ is much more powerful and closer to spaces (a field) which are otherwise used in mathematics. So it would be worth integrating arithmetic of $\mathbb{I}^*\mathbb{R}$ into a future standard for interval arithmetic. This should not be too complicated. A definition of arithmetic in $\mathbb{I}^*\mathbb{R}$ should just present the tables for the arithmetic operations (16 cases instead of nine in case of multiplication, for instance) and briefly mention how other concepts are extended to the new elements of $\mathbb{I}^*\mathbb{R}$ as level 1 operations and it should explain how the operations of $\mathbb{I}^*\mathbb{F}$ with directed roundings as level 2 operations are to be performed. A much more powerful mathematical structure makes this development very attractive. All this might lead to many new applications of interval arithmetic. Thus we present the following

Motion

The Basic Operations of Kaucher Arithmetic

Here we sketch the basic operations of Kaucher-arithmetic. The formulas look very similar to those for the corresponding operations of $\mathbb{IR}$. The operands, however, are no longer only intervals of real numbers. They now can be proper or improper intervals. Arithmetic in $\mathbb{I}^*\mathbb{R}$ is just an arithmetic for pairs of real numbers with many desirable properties. The result of an operation can or can not be a real interval. Arithmetic for intervals of $\mathbb{IR}$ can be performed by arithmetic of $\mathbb{I}^*\mathbb{R}$ if only operands are used where the second component is greater or equal to the first one. With these remarks the following formulas widely speak for themselves. The operands $[a_1, a_2], [b_1, b_2]$ are elements of $\mathbb{I}^*\mathbb{R}$.

Negation $-[a_1, a_2] = [-a_2, -a_1]$.

Addition $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$.

Subtraction $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$.

Multiplication $[a_1, a_2] * [b_1, b_2]$	$[b_1, b_2]$ $b_1 \leq 0 \wedge b_2 \leq 0$	$[b_1, b_2]$ $b_1 < 0 < b_2$	$[b_1, b_2]$ $b_1 \geq 0 \wedge b_2 \geq 0$	$[b_1, b_2]$ $b_2 < 0 < b_1$
$a_1 \leq 0 \wedge a_2 \leq 0$	$[a_2 * b_2, a_1 * b_1]$	$[a_1 * b_2, a_1 * b_1]$	$[a_1 * b_2, a_2 * b_1]$	$[a_2 * b_2, a_2 * b_1]$
$a_1 < 0 < a_2$	$[a_2 * b_1, a_1 * b_1]$	$[min(a_1 * b_2, a_2 * b_1), max(a_1 * b_1, a_2 * b_2)]$	$[a_1 * b_2, a_2 * b_2]$	0
$a_1 \geq 0 \wedge a_2 \geq 0$	$[a_2 * b_1, a_1 * b_2]$	$[a_2 * b_1, a_2 * b_2]$	$[a_1 * b_1, a_2 * b_2]$	$[a_1 * b_1, a_1 * b_2]$
$a_2 < 0 < a_1$	$[a_2 * b_2, a_1 * b_2]$	0	$[a_1 * b_1, a_2 * b_1]$	$[max(a_1 * b_1, a_2 * b_2), min(a_1 * b_2, a_2 * b_1)]$

Division $[a_1, a_2]/[b_1, b_2] = [a_1, a_2] * 1/[b_1, b_2], 1/[b_1, b_2] = [1/b_2, 1/b_1], 0 \notin [b_1, b_2]$.

Additive Inverse $inv[a_1, a_2] = [-a_1, -a_2]$.

Square Root $sqrt([a_1, a_2]) = [sqrt(a_1), sqrt(a_2)]$, defined iff $a_1 \geq 0 \wedge a_2 \geq 0$.

Meet $meet([a_1, a_2], [b_1, b_2]) = [max(a_1, b_1), min(a_2, b_2)]$.

Join $join([a_1, a_2], [b_1, b_2]) = [min(a_1, b_1)], max(a_2, b_2)]$.

Minimum $min([a_1,a_2],[b_1,b_2]) = [min(a_1,b_1), min(a_2,b_2)]$.

Maximum $max([a_1,a_2],[b_1,b_2]) = [max(a_1,b_1), max(a_2,b_2)]$.

Comparison Relations

$$[a_1,a_2] = [b_1,b_2] := (a_1 = b_1 \wedge a_2 = b_2),$$
$$[a_1,a_2] \subseteq [b_1,b_2] \vee [b_1,b_2] \supseteq [a_1,a_2] := (b_1 \le a_1 \wedge a_2 \le b_2),$$
$$[a_1,a_2] \le [b_1,b_2] \vee [b_1,b_2] \ge [a_1,a_2] := (a_1 \le b_1 \wedge a_2 \le b_2).$$

Membership

$$a \in [b_1,b_2] :\Leftrightarrow \begin{cases} b_1 \le a \le b_2, & \text{if } [b_1,b_2] \in \mathbb{IR} \\ b_2 \le a \le b_1, & \text{otherwise.} \end{cases}$$

In these formulas the operands are just pairs of real numbers. On a computer, however, only subsets of real numbers, floating-point numbers for instance, are representable. So the question remains how to approximate these formulas on a computer. For this purpose we define the outward rouding $\Diamond : \mathbb{I}^*\mathbb{R} \to \mathbb{I}^*\mathbb{F}$ by $\Diamond[a,b] = [\nabla a, \triangle b]$, i.e., the rounding $\Diamond$ rounds the first component of an element $[a,b] \in \mathbb{I}^*\mathbb{R}$ downwards and the second component upwards. With this rounding the result of an operation with two pairs $[a_1,a_2],[b_1,b_2] \in \mathbb{I}^*\mathbb{F}$ can be summarized by

$$[a_1,a_2] \mathbin{\Diamond\!\!\!\!\circ}\, [b_1,b_2] := \Diamond([a_1,a_2] \circ [b_1,b_2]), \text{ for } \circ \in \{+,-,*,/\} \text{ with } 0 \notin [b_1,b_2] \text{ for } \circ = /.$$

This is in full accordance with the arithmetic in $\mathbb{IR}$.