Standard for Modal Interval Arithmetic DRAFT

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Contents

Preface

This document is a standard for modal interval arithmetic. It is being developed in parallel to the standard for interval arithmetic by the IEEE P1788 Working Group; the goal is to be as close to P1788 as possible while at the same time introducing the features and benefits of modal interval arithmetic that may not be included in the final P1788 document.

Notation

$\varnothing$	Empty set
$\in$	Set membership (element-of)
$\cap$	Set intersection
$\cup$	Set union
$\setminus$	Set subtraction
$\mathbb{N}$	The set $\{0, 1, 2, 3, \ldots\}$ of natural numbers
$\mathbb{Z}$	The set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^*$	The set $\mathbb{R} \setminus \{0\}$ of non-zero real numbers
$\mathbb{R}_{\geq 0}$	The set $\{x \in \mathbb{R} : x \geq 0\}$ of non-negative real numbers
$\mathbb{R}_{>0}$	The set $\{x \in \mathbb{R} : x > 0\}$ of strictly positive real numbers
$\overline{\mathbb{R}}$	The set $\mathbb{R} \cup \{-\infty, +\infty\}$ of extended real numbers
$\mathbb{F}$	A finite subset of $\mathbb{R}$, e. g., IEEE 754 floating-point numbers
$\overline{\mathbb{F}}$	A finite set $\mathbb{F} \cup \{-\infty, +\infty\}$ of extended real numbers
$\equiv$	Equivalent to (definition)
$\neg$	Logical not
$\wedge$	Logical and
$\vee$	Logical or
$\Rightarrow$	Logical implication (if-then)
$\Leftrightarrow$	Logical equivalence (if and only if)
$\forall$	Universal quantifier (for all)
$\exists$	Existential quantifier (there exists)

Introduction

Modal interval analysis is the semantic and algebraic completion of classical interval analysis. Embedded within the modal interval analysis is all the power of classical interval analysis and more.

Classical interval analysis, as originally conceived by T. Sunaga [5] and later popularized by American scientist R. E. Moore [4], is the connective link that unites the borderline of pure mathematics and natural phenomena. For example, the representation of numerical quantities within a computer by means of a finite number of digits precludes the possibility to represent an irrational number such as $\sqrt{2}$. The concept of $\sqrt{2} = 1.4142\ldots$ cannot be formed in this case without assuming a sequence of nested intervals

$$[1, 2] \supseteq [1.4, 1.5] \supseteq [1.41, 1.42] \supseteq [1.414, 1.415] \supseteq [1.4142, 1.4143] \supseteq \cdots$$

that can be said to be associated with a rule for calculating the exact solution to within some finite tolerance of approximation. In this regard, Sunaga remarks the concept of an interval is more fundamental than that of a real number.

In classical interval analysis, the set $X = [a, b] = \{x : a \leq x \leq b\}$ is called an interval. This definition presupposes that a and b are real numbers such that $a \leq b$. If X and Y are intervals, the inclusion relation $X \subseteq Y$ is true if and only if $x \in X$ implies $x \in Y$, i. e.,

$$X \subseteq Y \quad \Leftrightarrow \quad (\forall x \in X)(\exists y \in Y) : x = y.$$

The inclusion relation for intervals forms a partially ordered set so that if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$. Inclusion $X \subseteq Y$ may also be written $Y \supseteq X$.

If $\circ$ is the arithmetic operation addition, subtraction, multiplication or division, the corresponding arithmetic operations on intervals are defined

$$X \circ Y \equiv \{x \circ y : x \in X, y \in Y\},$$

provided that $0 \notin Y$ in the case of division. If $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ are intervals, these operations can be expressed by the interval endpoints as

$$\begin{aligned}
X + Y &\equiv [x_1 + y_1, x_2 + y_2], \\
X - Y &\equiv [x_1 - y_2, x_2 - y_1], \\
XY &\equiv [\min(x_1y_1, x_2y_1, x_1y_2, x_2y_2), \max(x_1y_1, x_2y_1, x_1y_2, x_2y_2)], \\
X/Y &\equiv X * (1/Y) \equiv X * [1/y_2, 1/y_1].
\end{aligned}$$

A generalization to interval functions can also be made. If $f(x_1, x_2, ..., x_n)$ is a real function and $X = (X_1, X_2, ..., X_n)$ is an interval box such that the restriction of f to X is continuous, the so-called "united extension"

$$f(X) \equiv \{f(x_1, x_2, ..., x_n) : x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}$$

is a set containing the image of f for all elements of X.

Several important properties of the united extension come from calculus. By the nested intervals theorem, if $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m \supseteq \cdots$ are m nested interval boxes, $X_1 \cap X_2 \cap \cdots \cap X_m \neq \varnothing$ and there exists a point $\varphi \in \mathbb{R}^n$ that is an element of each X_k for $k = 1, 2, \ldots, m$. If the dimensions of each X_k decrease as $m \to \infty$, the X_k converge to φ. In this case, φ is a limit point of X_1 such that

$$\bigcap_{k=1,2,3,\ldots} f(X_k) = f(\varphi).$$

By the extreme value theorem, the united extension has unique minimal and maximal elements; and by the intermediate value theorem, the united extension is a mapping of X to the interval

$$f(X) \equiv [\min_{x \in X} f(x), \max_{x \in X} f(x)].$$

These properties lead to the Fundamental Theorem of Interval Analysis [1], which says an interval Y is an "outer" estimation of $f(X)$ when

$$f(X) \subseteq Y \quad \Leftrightarrow \quad (\forall x \in X)(\exists y \in Y) : f(x) = y.$$

This fundamental theorem can be viewed as the all-important link between the realms of pure mathematics and natural phenomena. For example, if $g(x) = \sqrt{x}$ and $X = [1, 2]$, the united extension

$$g(X) = [1, \sqrt{2}] = [1, 1.4142\ldots]$$

is not directly computable in the natural world when the interval endpoints are to be represented with only a finite number of digits. If x is a real number and the rounding operators $\bigtriangledown(x)$ and $\bigtriangleup(x)$ are approximations of x such that $\bigtriangledown(x) \leq x$ and $\bigtriangleup(x) \geq x$, then the "outer" rounding of an interval $[a, b]$ is defined $\mathrm{Out}([a, b]) = [\bigtriangledown(a), \bigtriangleup(b)]$. The property $[a, b] \subseteq \mathrm{Out}([a, b])$ is satisfied and, e. g., makes valid the implication

$$g(X) \subseteq \mathrm{Out}(g(X)) \quad \Rightarrow \quad [1, \sqrt{2}] \subseteq [1, 1.4143].$$

Outer estimations of the arithmetic operations are similarly defined, and lead to another important theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a rational function and E be an expression with a finite set of variables $x = (x_1, x_2, ..., x_n)$ such that f is the function computed by E. Let $F(X)$ be the computational program indicated by E when the variables are substituted by intervals $X = (X_1, X_2, ..., X_n)$ and each real operator of E is transformed into an outer estimation of a corresponding interval operator. This gives the relation

$$f(X) \subseteq F(X) \quad \Leftrightarrow \quad (\forall x \in X)(\exists y \in F(X)) : f(x) = y,$$

which is also known as the Fundamental Theorem of Interval Arithmetic.

The algebraic structure of classical interval arithmetic lacks some important properties. Interval subtraction is not the inverse of interval addition, and the same is true of interval multiplication and division. In general terms, for any interval $A = [a_1, a_2]$ such that $a_1 \neq a_2$, there exists no interval X such that the interval equations $A + X = [0, 0]$ or $A * X = [1, 1]$ are true, and for $A + X = B$ or $A * X = B$ none of the “solutions” $X = B - A$ or $X = B/A$ actually exist. This anomaly can be solved by adding certain elements, called the “improper” intervals, to the classic set of “proper” intervals identified by Sunaga and Moore. This process is known in abstract algebra as the Grothendieck group construction; it is, for example, how the abelian group $(\mathbb{Z}, +)$ of integers is constructed from the commutative cancellative monoid $(\mathbb{N}, +)$ of natural numbers. When applied to intervals, the resulting algebraic structure is an abelian group for the operation of addition, as well as for the multiplication of intervals not containing zero. This discovery is due mainly to E. Kaucher [3] but was also conceived earlier in a less developed form by Polish mathematician M. Warmus [6] [7].

Despite the better algebraic structure of the Kaucher interval arithmetic, it did not receive a lot of attention by many scientists. Perhaps the most likely reason was due to the abstract nature of the resulting interval system. The new elements of the Kaucher arithmetic (the improper intervals) were a purely algebraic construction, and this may have concealed their deeper meaning; it may not have been immediately clear or obvious how to extend the Fundamental Theorem of Interval Analysis, for example, to interval functions accepting improper intervals as arguments. In the meantime, many successful advances were being made in the field of classical interval analysis for finding solutions to scientifically important problems such as systems of nonlinear equations and global optimization.

Originally conceived by E. Gardenes et. al. [2], modal interval analysis defines in logical terms the semantic meaning for the improper intervals of Kaucher arithmetic; in so doing, the Fundamental Theorem of Interval Analysis is generalized to include interval functions that may also accept improper intervals as arguments. A similar generalization to the Fundamental Theorem of Interval Arithmetic is also defined.

As a consequence, the Fundamental Theorem of Interval Analysis appears naturally as a special case within the modal interval analysis; and since the algebraic structure of Kaucher arithmetic is embedded within the modal interval arithmetic, modal interval analysis may be viewed as the all-encompassing theory that unifies the algebraic completion of classical interval arithmetic with the set-theoretic formulation of Kaucher arithmetic.

Chapter 1

Mathematical Intervals

$$\overline{\mathbb{IR}} \equiv \{[a,b] : a,b \in \overline{\mathbb{R}}\} \setminus \{[-\infty,-\infty],[+\infty,+\infty]\}$$

is the set of intervals supported by this standard. An interval is therefore an ordered pair $[a,b]$ such that $a,b \in \overline{\mathbb{R}}$, the two pairs $[-\infty,-\infty]$ and $[+\infty,+\infty]$ excluded. Note that no restriction $a \leq b$ is required. The empty set ($\varnothing$) is not an element of $\overline{\mathbb{IR}}$ and is not an interval.

Definition 1 (Set-Theoretic Interpretation) *For any interval* $[a,b] \in \overline{\mathbb{IR}}$*,*

$$\mathrm{Set}([a,b]) \equiv \{x \in \mathbb{R} : \min(a,b) \leq x \leq \max(a,b)\}.$$

The notation $x \in [a,b]$ *may be used as an abbreviation for* $x \in \mathrm{Set}([a,b])$*. The empty set is not an interval, and* $\mathrm{Set}(\varnothing) \equiv \varnothing$*.*

A closed interval includes all of its limit points. Every $[a,b] \in \overline{\mathbb{IR}}$ is a closed interval. If both a and b are real numbers, the interval is bounded.

$$\mathbb{IR} \equiv \{[a,b] \in \overline{\mathbb{IR}} : a,b \in \mathbb{R}\}$$

is the subset of bounded intervals supported by this standard. If an interval is not bounded, then at least one of the endpoints is $-\infty$ or $+\infty$ and the interval still contains all of its limit points but not all of its endpoints. Intervals of the form

$$[-\infty,+\infty],\ [-\infty,b],\ [a,+\infty],\ [+\infty,b],\ [a,-\infty] \text{ and } [+\infty,-\infty]$$

are therefore understood to be unbounded. Despite the use of square brackets, infinity is never an element of any interval, and

$$[-\infty,-\infty] \text{ and } [+\infty,+\infty]$$

are by definition not elements of $\overline{\mathbb{IR}}$.

Other "natural" subsets of $\overline{\mathbb{IR}}$ are

$$\begin{aligned} I(\overline{\mathbb{R}}) &\equiv \{[a,b] \in \overline{\mathbb{IR}} : a \leq b\}, \\ I(\mathbb{R}) &\equiv \{[a,b] \in \mathbb{IR} : a \leq b\}, \\ I_D(\overline{\mathbb{R}}) &\equiv \{[a,b] \in \overline{\mathbb{IR}} : a \geq b\}, \\ I_D(\mathbb{R}) &\equiv \{[a,b] \in \mathbb{IR} : a \geq b\}, \\ I_R(\mathbb{R}) &\equiv \{[a,b] \in \mathbb{IR} : a = b\}, \end{aligned}$$

and the inclusion relations between these subsets are

$$I_R(\mathbb{R}) \subseteq I(\mathbb{R}) \subseteq I(\overline{\mathbb{R}}) \quad \text{and} \quad I_R(\mathbb{R}) \subseteq I_D(\mathbb{R}) \subseteq I_D(\overline{\mathbb{R}}).$$

An interval $[a,b] \in I(\overline{\mathbb{R}})$ is a "proper" interval; an interval $[a,b] \in I_D(\overline{\mathbb{R}})$ is an "improper" interval; and an interval $[a,b] \in I_R(\mathbb{R})$ is a "singleton" interval identified with an element of the real number line. $I(\mathbb{R})$ is the historically important set of "classic" intervals made popular by the works of Sunaga and Moore, and $\mathbb{IR}$ is the set of Kaucher intervals.

Definition 2 (Geometric Interpretation) *Elements of $\mathbb{IR}$ can be visualized as points in the $\mathbb{R}^2$ plane, where canonical abscissa and ordinate are defined respectively as the left and right bound of an interval $[a,b] \in \mathbb{IR}$, i. e.,*

$$\lambda([a,b]) \equiv a \quad \textit{and} \quad \rho([a,b]) \equiv b.$$

Remark 1 *The main theory of modal interval analysis is constructed from the set $\mathbb{IR}$ of closed and bounded Kaucher intervals; an "extended" modal interval analysis involving unbounded elements from the set $\overline{\mathbb{IR}}$ of closed intervals is also defined under a special set of circumstances. The main reason for the extension to $\overline{\mathbb{IR}}$ is because in digital computing one must often consider "overflow," a concept that will be dealt with in the sequel.*

1.1 Logical Semantics

Definition 3 (Predicate) *Within the context of this standard, a predicate is a boolean function of one or more real variables.*

Definition 4 (Proposition) *A proposition is a predicate wherein each real variable is universally or existentially quantified.*

Definition 5 (Modal Quantifier) *A real variable x is quantified by the modal quantifier Q on the interval $[a,b] \in \overline{\mathbb{IR}}$ by*

$$\mathrm{Q}(x,[a,b]) \equiv \begin{cases} \exists x \in \mathrm{Set}([a,b]) & \textit{if } a \leq b, \\ \forall x \in \mathrm{Set}([a,b]) & \textit{if } a \geq b. \end{cases}$$

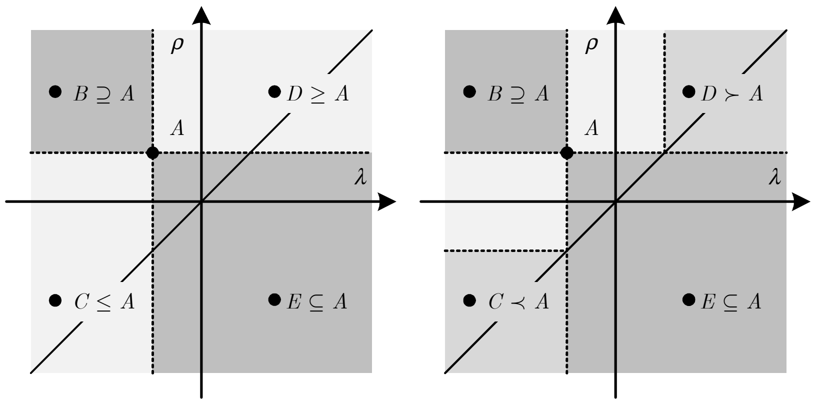

Figure 1.1: Comparison relations in $\mathbb{IR}$.

Remark 2 *If $A = [a, a]$ is a singleton interval and $P(x) : \mathbb{R} \to \{true, false\}$ is a predicate, then*

$$\mathrm{Q}(x, A)P(x) = (\exists x \in \mathrm{Set}(A))P(x) = (\forall x \in \mathrm{Set}(A))P(x) = P(a),$$

and in all cases the acceptance or rejection of the predicate P is decided by the value of $P(a)$.

Definition 6 (Modal Operators) *If $[a, b] \in \overline{\mathbb{IR}}$, the modal operators are*

$$\begin{aligned} \mathrm{Dual}([a,b]) &\equiv [b,a], \\ \mathrm{Prop}([a,b]) &\equiv [\min(a,b), \max(a,b)], \\ \mathrm{Impr}([a,b]) &\equiv [\max(a,b), \min(a,b)], \end{aligned}$$

and the corresponding modal quantifiers D, E *and* U *are*

$$\begin{aligned} \mathrm{D}(x,[a,b]) &\equiv \mathrm{Q}(x, \mathrm{Dual}([a,b])), \\ \mathrm{E}(x,[a,b]) &\equiv \mathrm{Q}(x, \mathrm{Prop}([a,b])), \\ \mathrm{U}(x,[a,b]) &\equiv \mathrm{Q}(x, \mathrm{Impr}([a,b])). \end{aligned}$$

1.2 Comparison Relations

Definition 7 (Inclusion) *If $A, B \in \overline{\mathbb{IR}}$, the inclusion ($\subseteq$) relation is*

$$A \subseteq B \quad \Leftrightarrow \quad \mathrm{D}(a, A)\, \mathrm{Q}(b, B) : a = b.$$

Definition 8 (Equality) *If $A, B \in \overline{\mathbb{IR}}$, the equality ($=$) relation is*

$$A = B \quad \Leftrightarrow \quad (A \subseteq B) \wedge (B \subseteq A).$$

Definition 9 (Less-or-Equal) *If $A, B \in \overline{\mathbb{IR}}$, the less-or-equal ($\leq$) and strictly less ($<$) relations are*

$$\begin{aligned} A \leq B &\Leftrightarrow (\mathrm{U}(a,A)\,\mathrm{E}(b,B) : a \leq b) \wedge (\mathrm{U}(b,B)\,\mathrm{E}(a,A) : a \leq b), \\ A < B &\Leftrightarrow (\mathrm{U}(a,A)\,\mathrm{E}(b,B) : a < b) \wedge (\mathrm{U}(b,B)\,\mathrm{E}(a,A) : a < b). \end{aligned}$$

Definition 10 (Interior) *If $A, B \in \overline{\mathbb{IR}}$, the interior ($\Subset$) relation is*

$$A \Subset B \Leftrightarrow (\mathrm{D}(a,A)\,\mathrm{Q}(b,B) : b < a) \wedge (\mathrm{D}(a,A)\,\mathrm{Q}(b,B) : a < b).$$

If A is proper and B is improper, the interior relation is never true.

Definition 11 (Precedes) *If $A, B \in \overline{\mathbb{IR}}$, the precedes ($\preccurlyeq$) and strictly precedes ($\prec$) relations are*

$$\begin{aligned} A \preccurlyeq B &\Leftrightarrow \mathrm{U}(a,A)\,\mathrm{U}(b,B) : a \leq b, \\ A \prec B &\Leftrightarrow \mathrm{U}(a,A)\,\mathrm{U}(b,B) : a < b. \end{aligned}$$

Lemma 1 (Programming of Relations) *If $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are elements of $\overline{\mathbb{IR}}$, the programming of comparison relations is*

$$\begin{aligned} A = B &\Leftrightarrow (a_1 = b_1) \wedge (a_2 = b_2), \\ A \leq B &\Leftrightarrow (a_1 \leq b_1) \wedge (a_2 \leq b_2), \\ A < B &\Leftrightarrow (a_1 \leqslant b_1) \wedge (a_2 \leqslant b_2), \\ A \subseteq B &\Leftrightarrow (b_1 \leq a_1) \wedge (a_2 \leq b_2), \\ A \Subset B &\Leftrightarrow (b_1 \leqslant a_1) \wedge (a_2 \leqslant b_2), \\ A \preccurlyeq B &\Leftrightarrow \max(a_1, a_2) \leq \min(b_1, b_2), \\ A \prec B &\Leftrightarrow \max(a_1, a_2) < \min(b_1, b_2), \end{aligned}$$

where $\leqslant$ is the same as $<$ except that $-\infty \leqslant -\infty$ and $+\infty \leqslant +\infty$ are true.

Several comparison relations are visualized in the $\mathbb{R}^2$ plane by Figure 1.1.

Definition 12 (Empty Relations) *The empty set ($\varnothing$) is not an interval, so any comparison relation involving the empty set is unordered. Any unordered comparison relation is false and its negation is true, e. g., $\varnothing = \varnothing$ is false and $\varnothing \neq \varnothing$ is true.*

1.3 Lattice Operators

Definition 13 (Meet and Join) *The algebraic structure $(\overline{\mathbb{IR}}, \subseteq)$ is a bounded lattice. For any elements $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of the lattice, the unique infimum and supremum are, respectively,*

$$\begin{aligned} \mathrm{meet}(A,B) &\equiv [\max(a_1, b_1), \min(a_2, b_2)], \\ \mathrm{join}(A,B) &\equiv [\min(a_1, b_1), \max(a_2, b_2)]. \end{aligned}$$

The bottom of the lattice is $[+\infty, -\infty]$ and the top of the lattice is $[-\infty, +\infty]$.

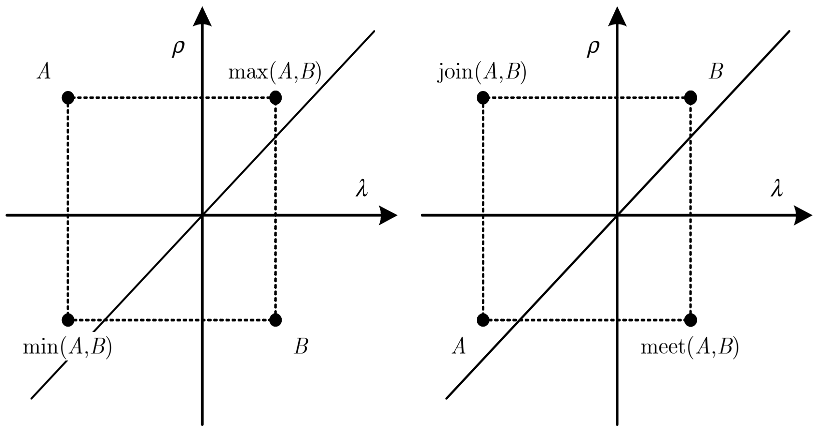

Figure 1.2: Lattice operators in $\mathbb{IR}$.

Definition 14 (Minimum and Maximum) *The algebraic structure*

$$(\overline{\mathbb{IR}} \cup \{[-\infty, -\infty], [+\infty, +\infty]\}, \leq)$$

is a bounded lattice. For any elements $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of the lattice, the unique infimum and supremum are, respectively,

$$\begin{aligned} \min(A, B) &\equiv [\min(a_1, b_1), \min(a_2, b_2)], \\ \max(A, B) &\equiv [\max(a_1, b_1), \max(a_2, b_2)]. \end{aligned}$$

The bottom of the lattice is $[-\infty, -\infty]$ and the top of the lattice is $[+\infty, +\infty]$, although these two elements of the lattice are not also elements of $\overline{\mathbb{IR}}$.

Lattice relations are visualized in the $\mathbb{R}^2$ plane by Figure 1.2.

1.4 Digital Intervals and Roundings

If $\mathbb{F} \subseteq \mathbb{R}$ is a digital scale for the real numbers and $\overline{\mathbb{F}} \equiv \mathbb{F} \cup \{-\infty, +\infty\}$,

$$\overline{\mathbb{IF}} \equiv \{[a, b] : a, b \in \overline{\mathbb{F}}\} \setminus \{[-\infty, -\infty], [+\infty, +\infty]\}$$

is the set of digital intervals supported by $\overline{\mathbb{F}}$. A digital interval is therefore an ordered pair $[a, b]$ such that $a, b \in \overline{\mathbb{F}}$, the two pairs $[-\infty, -\infty]$ and $[+\infty, +\infty]$ excluded. No restriction $a \leq b$ is required. The empty set ($\varnothing$) is not an element of $\overline{\mathbb{IF}}$ and

$$\mathbb{IF} \equiv \{[a, b] \in \overline{\mathbb{IF}} : a, b \in \mathbb{F}\}$$

is the subset of bounded digital intervals supported by $\mathbb{F}$.

Other subsets of $\overline{\mathbb{IF}}$ are

$$\begin{aligned} I(\overline{\mathbb{F}}) &\equiv \{[a,b] \in \overline{\mathbb{IF}} : a \leq b\}, \\ I(\mathbb{F}) &\equiv \{[a,b] \in \mathbb{IF} : a \leq b\}, \\ I_D(\overline{\mathbb{F}}) &\equiv \{[a,b] \in \overline{\mathbb{IF}} : a \geq b\}, \\ I_D(\mathbb{F}) &\equiv \{[a,b] \in \mathbb{IF} : a \geq b\}, \\ I_R(\mathbb{F}) &\equiv \{[a,b] \in \mathbb{IF} : a = b\}, \end{aligned}$$

and the inclusion relations between these subsets are

$$I_R(\mathbb{F}) \subseteq I(\mathbb{F}) \subseteq I(\overline{\mathbb{F}}) \quad \text{and} \quad I_R(\mathbb{F}) \subseteq I_D(\mathbb{F}) \subseteq I_D(\overline{\mathbb{F}}).$$

Definition 15 (Rounding Operators) *The rounding operators*

$$\bigtriangledown(x) : \overline{\mathbb{R}} \to \overline{\mathbb{F}} \quad \textit{and} \quad \bigtriangleup(x) : \overline{\mathbb{R}} \to \overline{\mathbb{F}}$$

are digital approximations of x such that the relations $\bigtriangledown(x) \leq x$ and $\bigtriangleup(x) \geq x$ are always true.

Definition 16 (Inner and Outer Roundings) *For $[a,b] \in \overline{\mathbb{IR}}$,*

$$\text{Inn}([a,b]) \equiv [\bigtriangleup(a), \bigtriangledown(b)] \quad \textit{and} \quad \text{Out}([a,b]) \equiv [\bigtriangledown(a), \bigtriangleup(b)]$$

are the "inner" and "outer" digital roundings, respectively, of $[a,b]$.

The inner and outer digital roundings are universally possible within the limits of any $\overline{\mathbb{F}}$ and satisfy the property

$$\text{Inn}([a,b]) \subseteq [a,b] \subseteq \text{Out}([a,b])$$

such that the equivalence

$$\text{Inn}([a,b]) \equiv \text{Dual}(\text{Out}(\text{Dual}([a,b])))$$

makes unnecessary the implementation of the inner rounding. This standard therefore requires conforming implementations only to support outer digital roundings.

Chapter 2

Interval Extensions

The main theory of modal interval analysis is constructed from the set $\mathbb{IR}$ of nonempty, closed and bounded intervals. In the sequel, a so-called "extended" modal interval analysis involving elements from the set $\overline{\mathbb{IR}}$ of nonempty and closed intervals will be dealt with.

2.1 Semantic Functions

If $X \in \mathbb{IR}^n$, then $X = (X_p, X_i)$ is the "component splitting" of X such that the proper components of X are $X_p = (X_{p_1}, X_{p_2}, \ldots, X_{p_k})$ and the improper components of X are $X_i = (X_{i_{k+1}}, X_{i_{k+2}}, \ldots, X_{i_n})$. The component splitting is important to many definitions and theorems of the modal interval analysis.

Definition 17 (*-Semantic Function) *If $X = (X_p, X_i) \in \mathbb{IR}^n$ is an interval component-splitting and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that the restriction of f to $\mathrm{Set}(X)$ is continuous, then*

$$f^*(X) \equiv [\min_{x_p \in X_p} \max_{x_i \in X_i} f(x_p, x_i), \max_{x_p \in X_p} \min_{x_i \in X_i} f(x_p, x_i)]$$

*is the *-semantic extension of f onto X.*

Definition 18 (-Semantic Function)** *If $X = (X_p, X_i) \in \mathbb{IR}^n$ is an interval component-splitting and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that the restriction of f to $\mathrm{Set}(X)$ is continuous, then*

$$f^{**}(X) \equiv [\max_{x_i \in X_i} \min_{x_p \in X_p} f(x_p, x_i), \min_{x_i \in X_i} \max_{x_p \in X_p} f(x_p, x_i)]$$

*is the **-semantic extension of f onto X.*

Remark 3 *If all of the X-components are proper intervals, then*

$$f^*(X) \equiv f^{**}(X) \equiv [\min_{x \in X} f(x), \max_{x \in X} f(x)],$$

which is equivalent to the united extension of classical interval analysis; and if all of the X-components are improper intervals,

$$f^*(X) \equiv f^{**}(X) \equiv [\max_{x \in X} f(x), \min_{x \in X} f(x)].$$

For any real function $f : \mathbb{R}^n \to \mathbb{R}$ and $X \in \mathbb{IR}^n$ such that $f^*(X)$ and $f^{**}(X)$ are both defined, the semantic functions have several important properties. The semantic functions are related by the equivalence

$$\mathrm{Dual}(f^*(X)) \equiv f^{**}(\mathrm{Dual}(X)),$$

as well as the inclusion relation

$$f^*(X) \subseteq f^{**}(X). \tag{2.1}$$

An important and special case of the relation (2.1) is when

$$f^*(X) = f^{**}(X),$$

and the function f is said to be "JM-commutable" for X (the JM in this name stands for "Join Meet"). Important examples of JM-commutable functions are one-variable continuous functions and every two-variable continuous function that is partially monotonic in a domain $I(\mathbb{R})^2$ like the arithmetic operators and power function. Semantic functions are inclusion isotonic, i. e., for $X, Y \in \mathbb{IR}^n$,

$$X \subseteq Y \Rightarrow (f^*(X) \subseteq f^*(Y), f^{**}(X) \subseteq f^{**}(Y)).$$

2.2 Semantic Theorems

For $X \in \mathbb{IR}^n$, the semantic functions $f^*(X)$ or $f^{**}(X)$ are exact semantic images of f on X. When all the X-components are proper, $f^*(X)$ and $f^{**}(X)$ correspond to the united extension of classical interval analysis and represent the exact image of the real function f on $\mathrm{Set}(X)$. If at least one component of X is improper, however, the semantic functions may not yield, without further thought, much clear meaning about the image of f on X.

Two key theorems uncover the meaning of the semantic functions in logical terms. The Fundamental Theorem of Interval Analysis from classical interval analysis appears as a special case of these key theorems; in the general case, a logic formula and corresponding inclusion relation for an "outer" estimation of $f^*(X)$ and an "inner" estimation of $f^{**}(X)$ are given, even if some or all components of X may be improper.

Theorem 1 (*-Semantic Theorem) *If $X = (X_p, X_i) \in \mathbb{IR}^n$ is an interval component-splitting and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that the restriction of f to $\mathrm{Set}(X)$ is continuous, then for $Y^* \in \mathbb{IR}$,*

$$f^*(X) \subseteq Y^* \Leftrightarrow \mathrm{U}(x_p, X_p)\, \mathrm{Q}(y, Y^*)\, \mathrm{E}(x_i, X_i) : f(x_p, x_i) = y.$$

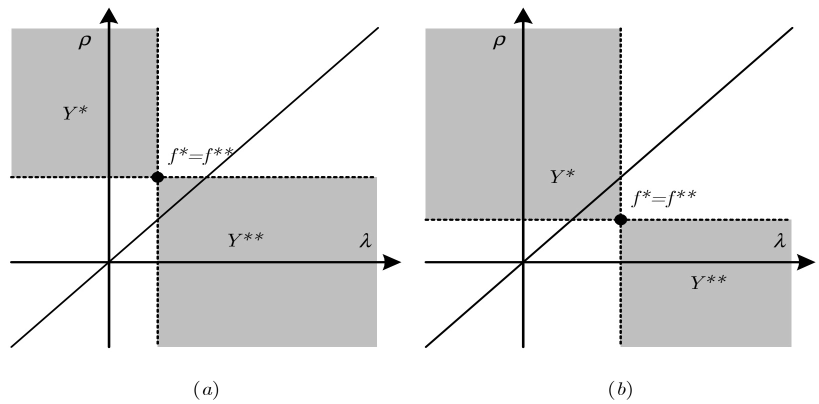

Figure 2.1: An illustration of "outer" and "inner" estimations Y^* and Y^{**}, respectively, of (2.4)–(2.7). All points in the plane above the $\lambda = \rho$ line are proper intervals and points below this line are improper intervals. Any point in a shaded region is a value of Y^* or Y^{**} that satisfies one of the corresponding semantics. Estimations for (2.2) are depicted on the left in (a) and estimations for (2.3) are depicted on the right in (b). If f^* is a proper interval, then Y^* is a proper interval. If f^* is an improper interval, then Y^* may be a proper or an improper interval. This is why classical interval analysis is the special case depicted in (a) where f^* and Y^* are proper intervals.

Theorem 2 (-Semantic Theorem)** *If $X = (X_p, X_i) \in \mathbb{IR}^n$ is an interval component-splitting and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that the restriction of f to $\mathrm{Set}(X)$ is continuous, then for $Y^{**} \in \mathbb{IR}$,*

$$Y^{**} \subseteq f^{**}(X) \Leftrightarrow \mathrm{U}(x_i, X_i)\,\mathrm{D}(y, Y^{**})\,\mathrm{E}(x_p, X_p) : f(x_p, x_i) = y.$$

Remark 4 *When all the X-components are proper intervals, $f^*(X)$ is a proper interval. No proper interval can be a subset of an improper interval, so Y^* must be proper for the *-semantic to be true. Under this assumption, the *-semantic simplifies to*

$$f^*(X) \subseteq Y^* \Leftrightarrow (\forall x \in X)(\exists y \in Y^*) : f(x) = y.$$

This is the Fundamental Theorem of Interval Analysis from classical interval analysis.

Example 1 *Consider the real function $f(x) = \sqrt{x}$ for the proper and improper intervals $[1, 4]$ and $[4, 1]$. The semantic functions are*

$$f^*([1,4]) = f^{**}([1,4]) = [1,2], \tag{2.2}$$

$$f^*([4,1]) = f^{**}([4,1]) = [2,1]. \tag{2.3}$$

The semantic theorems then give the inclusion relations between each semantic function (2.2)–(2.3) and an "outer" estimation

$$f^*([1,4]) \subseteq Y^* \Leftrightarrow (\forall x \in \mathrm{Set}([1,4]))\, \mathrm{Q}(y, Y^*) : f(x) = y, \tag{2.4}$$

$$f^*([4,1]) \subseteq Y^* \Leftrightarrow \mathrm{Q}(y, Y^*)(\exists x \in \mathrm{Set}([4,1])) : f(x) = y, \tag{2.5}$$

as well as an "inner" estimation

$$Y^{**} \subseteq f^{**}([1,4]) \Leftrightarrow \mathrm{D}(y, Y^{**})(\exists x \in \mathrm{Set}([1,4])) : f(x) = y, \tag{2.6}$$

$$Y^{**} \subseteq f^{**}([4,1]) \Leftrightarrow (\forall x \in \mathrm{Set}([4,1]))\, \mathrm{D}(y, Y^{**}) : f(x) = y. \tag{2.7}$$

In order to visualize the intervals Y^* and Y^{**} which satisfy (2.4)–(2.7), the semantic functions $f^* = f^{**}$ can be plotted in $\mathbb{R}^2$ as a point that divides the plane into four quadrants as depicted in Figure 2.1. The shaded regions are the intervals which satisfy one of the corresponding semantics.

Remark 5 *The function $f(x) = \sqrt{x}$ is JM-commutable; this means both the *- and **-semantics are true when*

$$f(X)^* = Y^* = Y^{**} = f(X)^{**}.$$

Remark 6 *The proper interval (2.2) cannot be a subset of an improper interval, so Y^* must be proper for the *-semantic to be true. Under this assumption, (2.4) may be expanded to*

$$f^*([1,4]) \subseteq Y^* \Leftrightarrow (\forall x \in [1,4])(\exists y \in Y^*) : f(x) = y.$$

This is the "outer" estimation of classical interval analysis. The interval (2.3), however, may be a subset of a proper or improper interval, so $\mathrm{Q}(y, Y^*)$ cannot be expanded in (2.5) until the value of Y^* is known. For example, if $Y^* = [1.9, 1.2]$, then (2.5) expands to

$$f^*([4,1]) \subseteq [1.9, 1.2] \Leftrightarrow (\forall y \in \mathrm{Set}([1.9, 1.2]))(\exists x \in \mathrm{Set}([4,1])) : f(x) = y;$$

but if $Y^* = [-1, 4]$, then (2.5) expands to

$$f^*([4,1]) \subseteq [-1,4] \Leftrightarrow (\exists y \in \mathrm{Set}([-1,4]))(\exists x \in \mathrm{Set}([4,1])) : f(x) = y.$$

Similar cases exist for (2.6)–(2.7). In all cases, the predictions made by the semantic theorems are true.

2.3 Syntactic Functions

Looking at the syntactic tree for a real function $f : \mathbb{R}^n \to \mathbb{R}$, where the nodes are operators, the leaves are variables, and branches define the domain of each operator, f can be operationally extended to a syntactic function $\mathbb{IR}^n \to \mathbb{IR}$ by using the computational program implicitly defined by the syntactic tree of the expression defining the function.

Definition 19 (Syntactic *-Function) *The syntactic *-function*

$$fR^*(X) : \mathbb{IR}^n \rightarrow \mathbb{IR}$$

*is defined by the computational program indicated by the syntactic tree of a real function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ when all of the real operators are transformed into their *-semantic extension.*

Definition 20 (Syntactic **-Function) *the syntactic **-function*

$$fR^{**}(X) : \mathbb{IR}^n \rightarrow \mathbb{IR}$$

*is defined by the computational program indicated by the syntactic tree of a real function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ when all of the real operators are transformed into their **-semantic extension.*

The syntactic functions are related by the equivalence

$$\mathrm{Dual}(fR^*(X)) \equiv fR^{**}(\mathrm{Dual}(X)),$$

and the syntactic functions are inclusion isotonic, i. e., for $X, Y \in \mathbb{IR}^n$,

$$X \subseteq Y \Rightarrow (fR^*(X) \subseteq fR^*(Y), fR^{**}(X) \subseteq fR^{**}(Y)).$$

When computing with digital intervals, the syntactic *-function uses an outer rounding computation and the syntactic **-function uses an inner rounding computation.

Definition 21 (Outer Rounding Computation) *The outer rounding computation*

$$\mathrm{Out}(fR^*(X))$$

is the function defined by the computational program of $fR^*(X)$ when the value of every X-component is replaced by its outer rounding and the exact value of every operator is replaced by its outer rounding.

Definition 22 (Inner Rounding Computation) *The inner rounding computation*

$$\mathrm{Inn}(fR^{**}(X))$$

is the function defined by the computational program of $fR^{**}(X)$ when the value of every X-component is replaced by its inner rounding and the exact value of every operator is replaced by its inner rounding.

Theorem 3 (Dual Computing Process) *The inner rounding of $fR^{**}(X)$ may be computed in terms of the outer rounding of $fR^*(X)$ by the equivalence*

$$\mathrm{Inn}(fR^{**}(X)) \equiv \mathrm{Dual}(\mathrm{Out}(fR^*(\mathrm{Dual}(X)))).$$

In practice, computations do not need an arithmetic supporting both the inner and outer roundings because of the dual computing process. This standard therefore requires conforming implementations only to support outer rounding computations.

Chapter 3

Interpretability and Optimality

If $X \in \mathbb{IR}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that the restriction of f to $\mathrm{Set}(X)$ is continuous, then for $Y^*, Y^{**} \in \mathbb{IR}$ the semantic theorems define an equivalence between a first-order logic formula, involving equalities relating the real function f to elements of X, and an inclusion relation involving Y^* or Y^{**}. If one of the semantic theorems verifies the relation

$$f^*(X) \subseteq Y^* \quad \text{or} \quad Y^{**} \subseteq f^{**}(X),$$

then f is said to be *interpretable*. Particularly, f may be qualified as *- or **-interpretable depending on which semantic is true. If both semantic theorems are true, thus verifying the JM-commutability of the relation

$$f(X)^* = Y^* = Y^{**} = f(X)^{**},$$

then f is said to be *optimal*.

3.1 Digital Computations

There are many theorems in the literature about the interpretability of syntactic functions. If $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that $f^*(X)$ and $f^{**}(X)$ are the semantic functions and $fR^*(X)$ and $fR^{**}(X)$ are the syntactic functions, the interpretable relations

$$f^*(X) \subseteq \mathrm{Out}(fR^*(X)) \quad \text{and} \quad \mathrm{Inn}(fR^{**}(X)) \subseteq f^{**}(X)$$

are true only under a specific set of circumstances given by the "theorems of interpretability." In the cases not directly covered by these theorems, the input X may be transformed according to certain rules so that the syntactic functions become interpretable by the so-called "theorems of coercion." An in-depth presentation of all these theorems is beyond the scope of this standard, but they may be found in the literature [2].

Remark 7 *When all the X-components are proper intervals, both $f^*(X)$ and* $\mathrm{Out}(fR^*(X))$ *are proper intervals such that the interpretable relation*

$$f^*(X) \subseteq \mathrm{Out}(fR^*(X))$$

is always true. This special case coincides with the Fundamental Theorem of Interval Arithmetic from classical interval analysis.

Chapter 4

Extended Modal Interval Analysis

This section introduces the concepts of extended modal interval analysis. The main theories of modal interval analysis are constructed on the premise that an interval extension is defined only when the restriction of a real function to an input domain is continuous, the input domain being a set of nonempty and bounded intervals. With extended modal interval analysis, semantic theorems and syntactic functions are extended to elements of $\overline{\mathbb{IR}}$ by taking into consideration continuity considerations, and an exception handling system based on the concept of *decorations* is used to reliably detect out-of-domain arguments and evaluation of non-continuous functions.

4.1 Continuity Considerations

The extension of a continuous real function to an interval operator coincides with the intermediate value theorem of calculus, which states the image of an interval on a continuous function is another interval. The meaning is two-fold, namely the logic formulas of both the *- and **-semantic theorems are necessary (in fact, one is necessary and the other is sufficient) for a correct definition of an interval operator. More specifically, an interval operator must be optimal.

Definition 23 (Operator) *If $X \in \overline{\mathbb{IR}}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that f is optimal for X, then f is an operator for X.*

Example 2 *In classical interval analysis, the extension of the continuous real function $f(x) = x^2$ to the interval $X = [-1, 3]$ is*

$$f(X) = \{x^2 : x \in [-1, 3]\} = [\min_{x\in[-1,3]} x^2, \max_{x\in[-1,3]} x^2] = [0, 9].$$

This coincides with the intermediate value theorem from calculus, which states the image of an interval on a continuous function is another interval. The *implications are two-fold, namely the semantic*

$$(\forall x \in [-1,3])(\exists y \in [0,9]) : y = x^2$$

and the semantic

$$(\forall y \in [0,9])(\exists x \in [-1,3]) : y = x^2$$

are both true. In the context of modal interval analysis, $f(X)$ is a syntactic operator since f is optimal on X.

Definition 24 (Empty Operator) *If $X \in \varnothing \cup \overline{\mathbb{IR}}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function such that f is not a syntactic operator for X, then the operation on X is the empty set.*

Example 3 *The extension of the non-continuous real function $f(x) = \text{floor}(x)$ to the interval $X = [-1,3]$, i. e.,*

$$f(X) = \{\text{floor}(x) : x \in [-1,3]\} = \{-1,0,1,2,3\}, \tag{4.1}$$

is not an interval and, therefore, it is not an interval extension. Even though minimal and maximal elements of (4.1) exist so that a unique interval

$$[\min_{x\in[-1,3]} \text{floor}(x), \max_{x\in[-1,3]} \text{floor}(x)] = [-1,3]$$

is defined that contains all possible values of (4.1), it is significantly important to recognize the semantic

$$(\forall x \in [-1,3])(\exists y \in [-1,3]) : y = \text{floor}(x)$$

is true, but the semantic

$$(\forall y \in [-1,3])(\exists x \in [-1,3]) : y = \text{floor}(x)$$

is false. So, on the interval $[-1,3]$ the floor function is an empty operator.

Example 4 *The extension of the real function $f(x) = \sqrt{x}$ to the interval $X = [-4,-1]$ is undefined and is not an interval, i. e.,*

$$f(X) = \{\sqrt{x} : x \in [-4,-1]\} = \varnothing.$$

Definition 25 (Promoted Operator) *If $X \in \overline{\mathbb{IR}}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function with natural domain $D_f \subseteq \mathbb{R}^n$ such that f is an empty operator for X but the relation $\text{Set}(X) \cap D_f \neq \varnothing$ is also true, then f may be promoted to a syntactic operator by further restricting the domain of f to $\text{Set}(X) \cap D_f$.*

Example 5 *The natural domain of the real function $f(x) = 1/x$ is the set $\mathbb{R}^*$ of non-zero real numbers. On the interval $X = [0,1]$, the function is an empty operator because $0 \in X$. However, if the domain of f is further restricted to $\text{Set}(X) \cap \mathbb{R}^* = (0,1]$, then the semantic*

$$(\forall x \in (0,1])(\exists y \in [1,+\infty]) : y = 1/x$$

and the semantic

$$(\forall y \in [1, +\infty])(\exists x \in (0, 1]) : y = 1/x$$

are both true, so the promoted function is a syntactic operator for X. However, if $X = [-1, 1]$ and $\mathrm{Set}(X) \cap D_f = [-1, 0) \cup (0, 1]$, then the semantic

$$(\forall x \in [-1, 0) \cup (0, 1])(\exists y \in [1, +\infty]) : y = 1/x$$

is true, but the semantic

$$(\forall y \in [1, +\infty])(\exists x \in [-1, 0) \cup (0, 1]) : y = 1/x$$

is false. So, on the interval $[-1, 1]$ the promoted function is still an empty operator.

4.2 Decorations

Decorations describe mathematical properties of the restriction of a real function to an interval domain. Decorations provide a framework for detecting exceptional conditions such as out-of-domain arguments or non-continuous functions.

$T(f, X)$	The restriction of f to $\mathrm{Set}(X)$ is...
`EIN`	defined and continuous on empty input
`DAC`	defined and continuous
`DEF`	defined
`GAP`	empty or nonempty
`NDF`	empty

$$\texttt{EIN} \subseteq \texttt{DAC} \subseteq \texttt{DEF} \subseteq \texttt{GAP} \supseteq \texttt{NDF} \supseteq \texttt{EIN}$$

$S(f, X)$	The restriction of f to $\mathrm{Set}(X)$ is...
`ein`	`EIN`
`dac`	`DAC` and not `EIN`
`def`	`DEF` and not `DAC`
`gap`	`GAP` and not (`DEF` or `NDF`)
`ndf`	`NDF` and not `EIN`

Definition 26 (Continuous Property) *If $X \in \varnothing \cup \overline{\mathbb{IR}}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function, then the continous property, denoted $C(f, X)$, is true if and only if the restriction of f to $\mathrm{Set}(X)$ is continuous. The empty set is not an interval, but the restriction of f to the empty set is continuous, i. e., $C(f, \varnothing)$ is true.*

Definition 27 (Domain Property) *If $X \in \varnothing \cup \overline{\mathbb{IR}}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a real function with natural domain $D_f \subseteq \mathbb{R}^n$, then the domain property is a pair of* *universally qualified propositions, denoted* $D(f,X)^+$ *and* $D(f,X)^-$*, that make opposite assertions about the set-membership of* X *in* D_f*, i. e.,*

$$\begin{aligned} D(f,X)^+ &\equiv \mathrm{U}(x,X) : (x \in D_f), \\ D(f,X)^- &\equiv \mathrm{U}(x,X) : \neg(x \in D_f). \end{aligned}$$

The empty set is not an interval, but for any f *the propositions* $D(f,\varnothing)^+$ *and* $D(f,\varnothing)^-$ *are always true.*

Definition 28 (Static Decorations) *If* $X \in \varnothing \cup \overline{\mathbb{IR}}^n$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *is a real function with natural domain* $D_f \subseteq \mathbb{R}^n$*, then* $S(f,X)$ *is a function that maps the* (f,X) *pair onto the set of static decorations*

$$\mathbb{D} \equiv \{\mathtt{ndf}, \mathtt{gap}, \mathtt{def}, \mathtt{dac}, \mathtt{ein}\}$$

according to the truth table:

$S(f,X)$	$D(f,X)^+$	$D(f,X)^-$	$C(f,X)$
`ein`	T	T	T
`dac`	T	F	T
`def`	T	F	F
`gap`	F	F	F
`ndf`	F	T	F

Static decorations partition the universe of all (f,X) *pairs into five disjoint sets, such that every* (f,X) *pair is associated with a static decoration that has the meaning:*

(f,X)	$X = \varnothing$	*The restriction of* f *to* Set(X) *is...*
`ein`	T	*defined and continuous on empty input*
`dac`	F	*defined and continuous*
`def`	F	*defined*
`gap`	F	*defined and not defined (information gap)*
`ndf`	F	*not defined*

4.3 Property Tracking

Decorations introduce the concept of a *decorated interval*, which is a pair (X,D) consisting of an interval $X \in \overline{\mathbb{IR}}$ and a decoration $D \in \mathbb{D}$, or a *decorated empty set*, which is a pair $(\varnothing, D)$. The union of decorated intervals and decorated empty sets is the set of *decorated elements*. The computation of a decorated element is modeled as a decorated *- or **-extension of a real function, and the computational process may be realized as a decorated syntactic *- or **-function.

Definition 29 (Tracking Decoration) *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is a real function and* $(X,D) \in (\varnothing \cup \overline{\mathbb{IR}}^n, \mathbb{D}^n)$ *are the decorated arguments of a semantic extension of*

f, then the tracking decoration $T(f,(X,D))$ is the greatest element by the linear quality order

$$\texttt{ndf} < \texttt{gap} < \texttt{def} < \texttt{dac} < \texttt{ein}$$

that is less-than or equal to all elements of a set formed by the union of the static decoration $S(f,X)$ and the components of D, i. e.,

$$T(f,(X,D)) \equiv \inf\{S(f,X), D_1, D_2, \ldots, D_n\}.$$

Definition 30 (Decorated *-Function) *If $f^* : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$ is the *-extension of a real function f and $(X,D) \in (\varnothing \cup \overline{\mathbb{IR}}^n, \mathbb{D}^n)$ are the decorated arguments of f^*, then the decorated *-function of f^* is*

$$(f^*(X), T(f,(X,D))). \tag{4.2}$$

*If f is an empty operator, then f is promoted to a syntactic operator, if possible, and the first component of (4.2) is the *-extension of the promoted operation and the second component of (4.2) is the tracking decoration of the unpromoted operation.*

Definition 31 (Decorated **-Extension) *If $f^{**} : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$ is the **-extension of a real function f and $(X,D) \in (\varnothing \cup \overline{\mathbb{IR}}^n, \mathbb{D}^n)$ are the decorated arguments of f^{**}, then the decorated **-extension of f^{**} is*

$$(f^{**}(X), T(f,(X,D))). \tag{4.3}$$

*If f is an empty operator, then f is promoted to a syntactic operator, if possible, and the first component of (4.3) is the **-extension of the promoted operation and the second component of (4.3) is the tracking decoration of the unpromoted operation.*

Definition 32 (Decorated Syntactic *-Extension) *If $fR^* : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$ is the syntactic *-extension of a real function f and $(X,D) \in (\varnothing \cup \overline{\mathbb{IR}}^n, \mathbb{D}^n)$ are the decorated arguments of fR^*, then the decorated syntactic *-extension of fR^*, denoted*

$$(fR^*(X), T(fR^*,(X,D))),$$

*is defined by the computational program indicated by the syntactic tree of fR^* when all of the operators are transformed into their decorated *-extensions.*

Definition 33 (Decorated Syntactic **-Extension) *If $fR^{**} : \overline{\mathbb{IR}}^n \to \overline{\mathbb{IR}}$ is the syntactic **-extension of a real function f and $(X,D) \in (\varnothing \cup \overline{\mathbb{IR}}^n, \mathbb{D}^n)$ are the decorated arguments of fR^{**}, then the decorated syntactic **-extension of fR^{**}, denoted*

$$(fR^{**}(X), T(fR^{**},(X,D))),$$

*is defined by the computational program indicated by the syntactic tree of fR^{**} when all of the operators are transformed into their decorated **-extensions.*

4.4 Promotions and Forgetful Operators

If $(X, D) \in (\varnothing \cup \overline{\mathbb{IR}}, \mathbb{D})$ is a decorated element, then a *bare interval*, a *bare empty set* or a *bare decoration* may be extracted from (X, D) by a *forgetful operator*, which throws away either the X or the D component of the decorated element (X, D).

By convention, a bare interval, bare empty set or bare decoration is just an interval, empty set or decoration, respectively, except when the bareness is to be emphasized. The union of bare intervals, bare empty set and bare decorations is the set of *bare elements*. A conforming implementation may permit the following promotions of bare elements:

- a bare interval $X \in \overline{\mathbb{IR}}$ may promote to a decorated interval $(X, \texttt{dac})$;
- a bare empty set may promote to a decorated empty set $(\varnothing, \texttt{ein})$; and
- a bare decoration $D \in \mathbb{D}$ may promote to a decorated empty set $(\varnothing, D)$.

If some or all of the arguments of an operation are bare elements, then the bare elements are implicitly promoted to decorated elements and the operation is then performed on the decorated elements. The result of the operation is a decorated element $(X, D) \in (\varnothing \cup \overline{\mathbb{IR}}, \mathbb{D})$, and a conforming implementation may choose to return (as if by an implicit use of a forgetful operator) only the X or the D component of the result.

Chapter 5

Required Operations

Conforming implementations shall provide the modal operators Dual, Prop, and Impr in Definition 6; the comparison relations in Section 1.2; and the lattice operators in Section 1.3.

5.1 Arithmetic Operations

Conforming implementations shall provide the arithmetic operations:

Name	Real function	Domain	Range
negate(x)	$-x$	$\mathbb{R}$	$\mathbb{R}$
recip(x)	$1/x$	$\mathbb{R}^*$	$\mathbb{R}^*$
add(x, y)	$x + y$	$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}$
sub(x, y)	$x - y$	$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}$
mul(x, y)	xy	$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}$
div(x, y)	x/y	$\mathbb{R} \times \mathbb{R}^*$	$\mathbb{R}$
abs(x)	$\|x\|$	$\mathbb{R}$	$\mathbb{R}_{\geq 0}$
sgn(x)	Sign of x	$\mathbb{R}$	$\{-1, 0, 1\}$
sqr(x)	x^2	$\mathbb{R}$	$\mathbb{R}_{\geq 0}$
sqrt(x)	$\sqrt{x}$	$\mathbb{R}_{\geq 0}$	$\mathbb{R}_{\geq 0}$
ceil(x)	$\lceil x \rceil$	$\mathbb{R}$	$\mathbb{Z}$
floor(x)	$\lfloor x \rfloor$	$\mathbb{R}$	$\mathbb{Z}$
nint(x)	Integer nearest to x	$\mathbb{R}$	$\mathbb{Z}$
trunc(x)	Integer truncation of x	$\mathbb{R}$	$\mathbb{Z}$
log(x)	$\ln x$	$\mathbb{R}_{>0}$	$\mathbb{R}$
log2(x)	$\log_2 x$	$\mathbb{R}_{>0}$	$\mathbb{R}$
log10(x)	$\log_{10} x$	$\mathbb{R}_{>0}$	$\mathbb{R}$
exp(x)	e^x	$\mathbb{R}$	$\mathbb{R}_{>0}$
exp2(x)	2^x	$\mathbb{R}$	$\mathbb{R}_{>0}$
exp10(x)	10^x	$\mathbb{R}$	$\mathbb{R}_{>0}$
pow(x, y)	$e^{y \ln(x)}$	$\mathbb{R}_{>0} \times \mathbb{R}$	$\mathbb{R}_{>0}$
cos(x)	Cosine	$\mathbb{R}$	$[-1, 1]$
sin(x)	Sine	$\mathbb{R}$	$[-1, 1]$
tan(x)	Tangent	$\mathbb{R} \setminus \{\pi\mathbb{Z} + \pi/2\}$	$\mathbb{R}$
acos(x)	Inverse cosine	$[-1, 1]$	$[0, \pi]$
asin(x)	Inverse sine	$[-1, 1]$	$[-\pi/2, \pi/2]$
atan(x)	Inverse tangent	$\mathbb{R}$	$(-\pi/2, \pi/2)$
cosh(x)	Hyperbolic cosine	$\mathbb{R}$	$[1, \infty)$
sinh(x)	Hyperbolic sine	$\mathbb{R}$	$\mathbb{R}$
tanh(x)	Hyperbolic tangent	$\mathbb{R}$	$(-1, 1)$
acosh(x)	Inverse hyperbolic cosine	$[1, \infty)$	$\mathbb{R}_{\geq 0}$
asinh(x)	Inverse hyperbolic sine	$\mathbb{R}$	$\mathbb{R}$
atanh(x)	Inverse hyperbolic tangent	$(-1, 1)$	$\mathbb{R}$

Each real function has a natural domain and range that defines the corresponding interval operator.

5.2 Constructors

For $a, b \in \overline{\mathbb{R}}$ such that $[a, b] \in \overline{\mathbb{IR}}$, and for $\mathtt{dec} \in \mathbb{D}$, conforming implementations shall provide the constructors:

Name	Definition	Description
$\mathtt{construct}(\mathtt{a}, \mathtt{b})$	$([\mathtt{a}, \mathtt{b}], \mathtt{dac})$	Construct a decorated interval
$\mathtt{construct}(\mathtt{a}, \mathtt{b}, \mathtt{dec})$	$([\mathtt{a}, \mathtt{b}], \mathtt{dec})$	Construct a decorated interval
$\mathtt{construct}(\mathtt{dec})$	$(\varnothing, \mathtt{dec})$	Construct a decorated empty set

There are no constructors for bare elements, but a conforming implementation may choose to return (as if by an implicit use of a forgetful operator) only the interval, empty set or decoration portion of a constructor.

5.3 Interval Measurement Functions

For any interval $[a, b] \in \overline{\mathbb{IR}}$, conforming implementations shall provide the measurement functions:

Name	Definition	Description
$\mathtt{inf}([\mathtt{a}, \mathtt{b}])$	$\mathtt{a}$	Greatest lower bound
$\mathtt{sup}([\mathtt{a}, \mathtt{b}])$	$\mathtt{b}$	Least upper bound
$\mathtt{mid}([\mathtt{a}, \mathtt{b}])$	$(\mathtt{a} + \mathtt{b})/2$	Midpoint
$\mathtt{rad}([\mathtt{a}, \mathtt{b}])$	$(\mathtt{b} - \mathtt{a})/2$	Radius
$\mathtt{wid}([\mathtt{a}, \mathtt{b}])$	$\lvert\mathtt{b} - \mathtt{a}\rvert$	Width
$\mathtt{mig}([\mathtt{a}, \mathtt{b}])$	$\inf\{\lvert x\rvert : x \in \mathrm{Set}([\mathtt{a}, \mathtt{b}])\}$	Mignitude
$\mathtt{mag}([\mathtt{a}, \mathtt{b}])$	$\sup\{\lvert x\rvert : x \in \mathrm{Set}([\mathtt{a}, \mathtt{b}])\}$	Magnitude

All of the measurement functions may return $+\infty$ or $-\infty$ depending on the interval argument. The radius function returns a negative radius for any improper interval with endpoints $a > b$. Midpoint is undefined for intervals $[-\infty, +\infty]$ and $[+\infty, -\infty]$.

5.4 Classification Functions

For any bare element $X \in \varnothing \cup \mathbb{D} \cup \overline{\mathbb{IR}}$, conforming implementations shall provide the classification functions:

Name	Definition	Description
$\mathtt{isProper}(\mathtt{X})$	$\mathtt{X} \in I(\overline{\mathbb{R}})$	Is X a proper interval?
$\mathtt{isImproper}(\mathtt{X})$	$\mathtt{X} \in I_i(\overline{\mathbb{R}})$	Is X an improper interval?
$\mathtt{isPoint}(\mathtt{X})$	$\mathtt{X} \in I_s(\overline{\mathbb{R}})$	Is X a singleton interval?
$\mathtt{isEmpty}(\mathtt{X})$	$\mathtt{X} \in \varnothing \cup \mathbb{D}$	Is X an empty set or decoration?

The same classification functions may be provided for any decorated element $(X, D) \in (\varnothing \cup \overline{\mathbb{IR}}, \mathbb{D})$ by examining only the X component of (X, D).

Appendix A

Groups and Monoids

The algebraic system $(\mathbb{R}, +)$ is an abelian group. For any $a \in \mathbb{R}$, the opposite (inverse element) of a is

$$\mathrm{Opp}(a) \equiv -a. \tag{A.1}$$

$\mathrm{Opp}(a)$ is an element of $\mathbb{R}$, and this satisfies the group axiom

$$a + \mathrm{Opp}(a) = a + (-a) = 0. \tag{A.2}$$

The algebraic system $(\mathbb{R}_{\geq 0}, +)$ is a commutative cancellative monoid, and for any $a \in \mathbb{R}_{\geq 0}$ which is not also an identity element of $\mathbb{R}_{\geq 0}$,

$$\mathrm{Opp}(a) \notin \mathbb{R}_{\geq 0},$$

as depicted in Figure A.1. Similarly, the set of "classic" intervals $I(\mathbb{R})$ is a subset of $\mathbb{IR}$. The algebraic system $(I(\mathbb{R}), +)$ is a commutative cancellative monoid, and for any $A \in I(\mathbb{R})$ which is not a singleton,

$$\mathrm{Opp}(A) \notin I(\mathbb{R}).$$

In abstract algebra, an abelian group may be constructed from a commutative cancellative monoid in the best possible way via the Grothendieck group construction. This is how $(\mathbb{R}, +)$ may be constructed from $(\mathbb{R}_{\geq 0}, +)$. This is also how $(\mathbb{IR}, +)$ may be constructed from $(I(\mathbb{R}), +)$, giving for any interval $[a, b] \in \mathbb{IR}$ the definitions

$$-([a,b]) \equiv [-b, -a], \tag{A.3}$$

$$\mathrm{Dual}([a,b]) \equiv [b, a], \tag{A.4}$$

$$\mathrm{Opp}([a,b]) \equiv [-a, -b] = -\,\mathrm{Dual}([a,b]), \tag{A.5}$$

as depicted in Figure A.2, such that for all $A, \mathrm{Opp}(A) \in \mathbb{IR}$

$$A + \mathrm{Opp}(A) = A + (-\,\mathrm{Dual}(A)) = [0, 0]. \tag{A.6}$$

Thus $(\mathbb{IR}, +)$ is an abelian group, and for any $A, B, X \in \mathbb{IR}$ the equation $A + X = B$ has the unique solution $X = B + \mathrm{Opp}(A)$. This is the beginning of Kaucher arithmetic [**?**], which is the algebraic completion of classical interval arithmetic [**?**], [**?**].

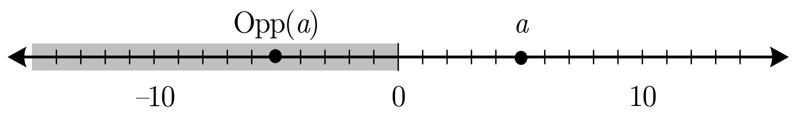

Figure A.1: The real number line. The shaded region represents the negative numbers which do not belong to the non-negative subset $\mathbb{R}_{\geq 0}$ of the reals. From a geometric perspective, the algebraic system $(\mathbb{R}_{\geq 0}, +)$ is not a group because Opp(a) is in the shaded region.

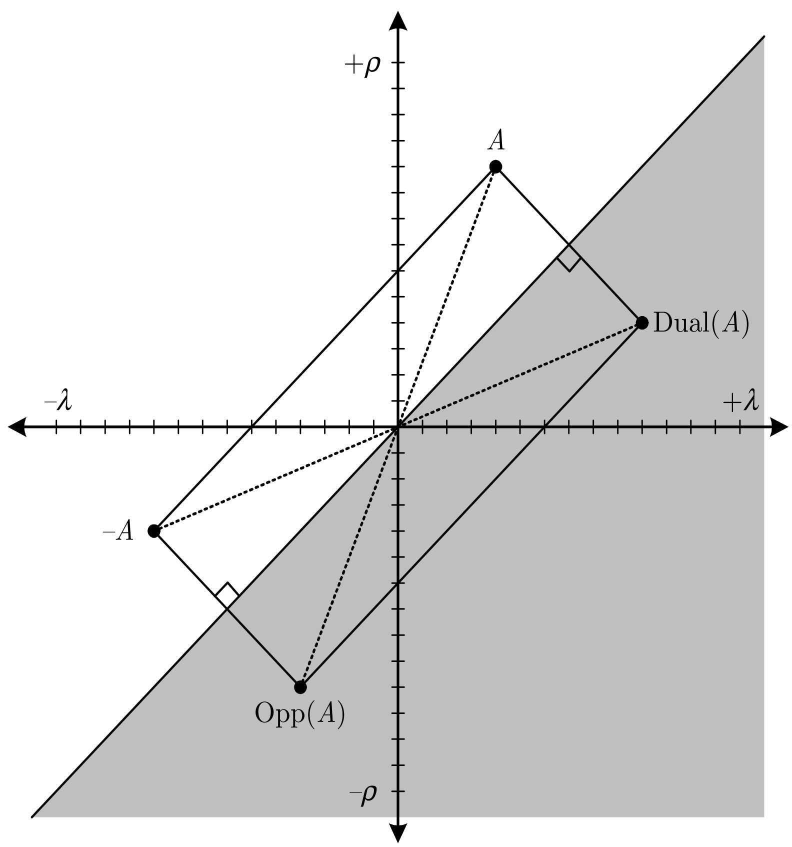

Figure A.2: $\mathbb{IR}$ is isomorphic to the $\mathbb{R}^2$ plane; and canonical abscissa and ordinate are defined respectively as the left and right bound of an interval $[a, b] \in \mathbb{IR}$, i. e., $\lambda([a, b]) = a$ and $\rho([a, b]) = b$. The shaded region represents the intervals $[a, b] \in \mathbb{IR}$ with $a > b$; these intervals do not belong to $I(\mathbb{R})$, the set of "classic" intervals, which are depicted as the region on or above the $\lambda = \rho$ line. In the analogy to $(\mathbb{R}_{\geq 0}, +)$, the algebraic system $(I(\mathbb{R}), +)$ is not a group since Opp(A) is in the shaded region.

Acknowledgements

The back matter often includes one or more of an index, an afterword, acknowledgements, a bibliography, a colophon, or any other similar item. In the back matter, chapters do not produce a chapter number, but they are entered in the table of contents. If you are not using anything in the back matter, you can delete the back matter TEX field and everything that follows it.

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