Motion divPair

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Following motion 5 the basic arithmetic operations are defined as powerset operations.

$$\mathbf{a} \bullet \mathbf{b} = \{a \circ b | a \in \mathbf{a}, b \in \mathbf{b}, \text{if it is defined}\}$$

Only for the division $\mathbf{a}/\mathbf{b}$ when $\mathbf{b}$ contains zero, the resulting sets are not necessarily intervals, there may be 2 semi-infinite closed or open intervals. We have 4 choices

(a) raise exception, terminate program

(b) split $\mathbf{b}$ in 2 and perform 2 divisions (motion 5)

(c) return a representation of the pair of unbounded intervals describing the exact solution set

(d) return the convex hull of the exact solution set; this is the P1788 division evaluation

Whereas (a) does not correspond to the paradigm of exception free computing (b) and (c) have the problem that the result is not an interval.

Hence (d) is the choice we have already voted for.

new operations

But especially for the interval Newton method, we want a division that computes two half bounded, disjoint intervals. This is also useful for the reverse multiplication operation `mulRev`, for which the 2 output intervals form the solution set of the equation $b * x = a$

$$C = \{x \in \mathbb{R} | \exists b \in \mathbf{b}, \exists a \in \mathbf{a}, b * x = a\} \tag{1}$$

The information returned by mulRevPair can be obtained by two ternary mulRev operations:

$mulRev(b, a, (-\infty, 0])$ and $mulRev(b, a, [0, +\infty))$

We want P1788 to provide a function that keeps the 2 quotients together.

We, therefore, move that P1788 shall have a version of the division and of the reverse multiplication operation, each of which returns 2 intervals. If the solution set consists of only 1 interval, the second output will be the empty set.

$$
\begin{aligned}
divPair : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} &\to \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} && (2)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\emptyset,\emptyset) \quad \text{if} \quad ((\mathbf{a}=\emptyset)\vee(\mathbf{a}=\emptyset)) && (3)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\overline{a}/\overline{b}],[\overline{a}/\underline{b},\infty]) \quad \text{if} \quad (\mathbf{a}<0)\wedge(0\subset\mathbf{b}) && (4)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\underline{a}/\underline{b}],[\underline{a}/\overline{b},\infty]) \quad \text{if} \quad (\mathbf{a}>0)\wedge(0\subset\mathbf{b}) && (5)\\
(\mathbf{a},\mathbf{b}) &\mapsto (0,\emptyset) \quad \text{if} \quad ((\mathbf{a}=0)\wedge(0\subseteq\mathbf{b})) && (6)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\emptyset,\emptyset) \quad \text{if} \quad ((\mathbf{b}=0)\wedge(0\subseteq\mathbf{a})) && (7)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\mathbf{a}/\mathbf{b},\emptyset) \quad \text{otherwise} && (8)
\end{aligned}
$$

$$
\begin{aligned}
mulRevPair : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} &\to \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} && (9)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\emptyset,\emptyset) \quad \text{if} \quad ((\mathbf{a}=\emptyset)\vee(\mathbf{a}=\emptyset)) && (10)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\overline{a}/\overline{b}],[\overline{a}/\underline{b},\infty]) \quad \text{if} \quad (\mathbf{a}<0)\wedge(0\subset\mathbf{b}) && (11)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\underline{a}/\underline{b}],[\underline{a}/\overline{b},\infty]) \quad \text{if} \quad (\mathbf{a}>0)\wedge(0\subset\mathbf{b}) && (12)\\
(\mathbf{a},\mathbf{b}) &\mapsto ([-\infty,\infty],\emptyset) \quad \text{if} \quad ((\mathbf{a}=0)\wedge(0\subseteq\mathbf{b}))\vee((\mathbf{b}=0)\wedge(0\subseteq\mathbf{a})) && (13)\\
(\mathbf{a},\mathbf{b}) &\mapsto (\mathbf{a}/\mathbf{b},\emptyset) \quad \text{otherwise} && (14)
\end{aligned}
$$

Note that we return a representation whose mapping to the two intervals is implementation defined. For infinite bounds use the general rules for infinities, but with $\infty/\infty = \infty$.

In this paper 0 denotes the point interval $[0,0]$.

Table 1 illustrates the usual forward division with hull (first line) returning one interval and the reverse multiplication without hull(second line) returning to disjoint intervals. Both are identical as long as the operands do not contain 0.

As a division divPair shall have a version for decorated intervals setting the local decoration to *trv*, if the denominator contains 0, and to *dac* or *com*, if not. mulRevPair has no decorated version, because it is a reverse operation.

	Div / **rev**	1 $\mathbf{b} = 0$	2 $\mathbf{b} \prec 0$	3 $\overline{b} = 0$	4 $0 \subset \mathbf{b}$	5 $0 = \underline{b}$	6 $0 \prec \mathbf{b}$
1	$\mathbf{a} = 0$	$\emptyset$ $[-\infty, +\infty]$	$[0,0]$ $[0,0]$	$[0,0]$ $[-\infty, +\infty]$	$[0,0]$ $[-\infty, +\infty]$	$[0,0]$ $[-\infty, +\infty]$	$[0,0]$ $[0,0]$
2	$\mathbf{a} \prec 0$	$\emptyset$ $\emptyset$	$[\overline{a} \bigtriangledown \underline{b}, \underline{a} \bigtriangleup \overline{b}]$ $[\overline{a} \bigtriangledown \underline{b}, \underline{a} \bigtriangleup \overline{b}]$	$[\overline{a} \bigtriangledown \underline{b}, +\infty]$ $[\overline{a} \bigtriangledown \underline{b}, +\infty]$	$[-\infty, +\infty]$ $(-\infty, \overline{a} \bigtriangleup \overline{b}]$ $\cup[\overline{a} \bigtriangledown \underline{b}, +\infty)$	$[-\infty, \overline{a} \bigtriangleup \overline{b}]$ $[-\infty, \overline{a} \bigtriangleup \overline{b}]$	$[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \overline{b}]$ $[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \overline{b}]$
3	$\overline{a} = 0$	$\emptyset$ $[-\infty, +\infty]$	$[\overline{a} \bigtriangledown \underline{b}, \underline{a} \bigtriangleup \overline{b}]$ $[\overline{a} \bigtriangledown \underline{b}, \underline{a} \bigtriangleup \overline{b}]$	$[\overline{a} \bigtriangledown \underline{b}, +\infty]$ $[-\infty, +\infty]$	$[-\infty, +\infty]$ $[-\infty, +\infty]$	$[-\infty, \overline{a} \bigtriangleup \overline{b}]$ $[-\infty, +\infty]$	$[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \overline{b}]$ $[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \overline{b}]$
4	$0 \subset \mathbf{a}$	$\emptyset$ $[-\infty, +\infty]$	$[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \overline{b}]$ $[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \overline{b}]$	$[-\infty, +\infty]$ $[-\infty, +\infty]$	$[-\infty, +\infty]$ $[-\infty, +\infty]$	$[-\infty, +\infty]$ $[-\infty, +\infty]$	$[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \underline{b}]$ $[\underline{a} \bigtriangledown \underline{b}, \overline{a} \bigtriangleup \underline{b}]$
5	$0 = \underline{a}$	$\emptyset$ $[-\infty, +\infty]$	$[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \underline{b}]$ $[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \underline{b}]$	$[-\infty, \underline{a} \bigtriangleup \underline{b}]$ $[-\infty, +\infty]$	$[-\infty, +\infty]$ $[-\infty, +\infty]$	$[\underline{a} \bigtriangledown \overline{b}, +\infty]$ $[-\infty, +\infty]$	$[\underline{a} \bigtriangledown \overline{b}, \overline{a} \bigtriangleup \underline{b}]$ $[\underline{a} \bigtriangledown \overline{b}, \overline{a} \bigtriangleup \underline{b}]$
6	$0 \prec \mathbf{a}$	$\emptyset$ $\emptyset$	$[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \underline{b}]$ $[\overline{a} \bigtriangledown \overline{b}, \underline{a} \bigtriangleup \underline{b}]$	$[-\infty, \underline{a} \bigtriangleup \underline{b}]$ $[-\infty, \underline{a} \bigtriangleup \underline{b}]$	$[-\infty, +\infty]$ $(-\infty, \underline{a} \bigtriangleup \underline{b}]$ $\cup[\underline{a} \bigtriangledown \overline{b}, +\infty)$	$[\underline{a} \bigtriangledown \overline{b}, +\infty]$ $[\underline{a} \bigtriangledown \overline{b}, +\infty]$	$[\underline{a} \bigtriangledown \overline{b}, \overline{a} \bigtriangleup \underline{b}]$ $[\underline{a} \bigtriangledown \overline{b}, \overline{a} \bigtriangleup \underline{b}]$

Table 1: Lookup table for division and reverse multiplication